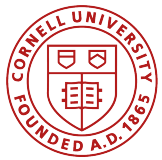
Convex Hulls in Geodesic Spaces of Nonpositive Curvature

Ariel Goodwin, Center for Applied Mathematics

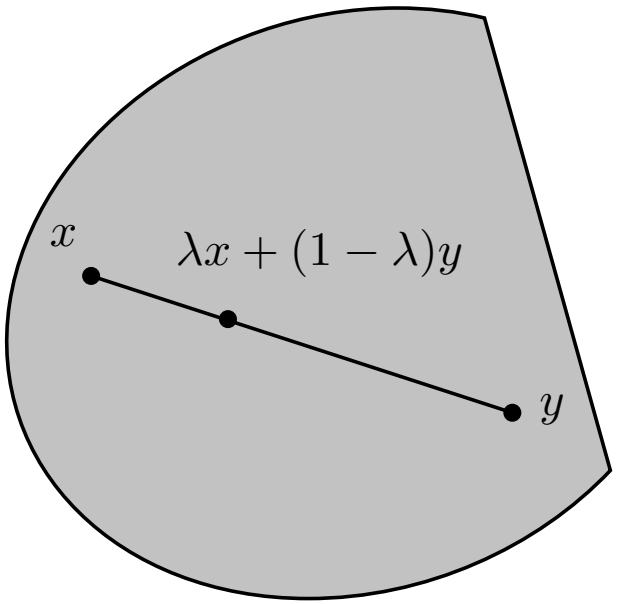


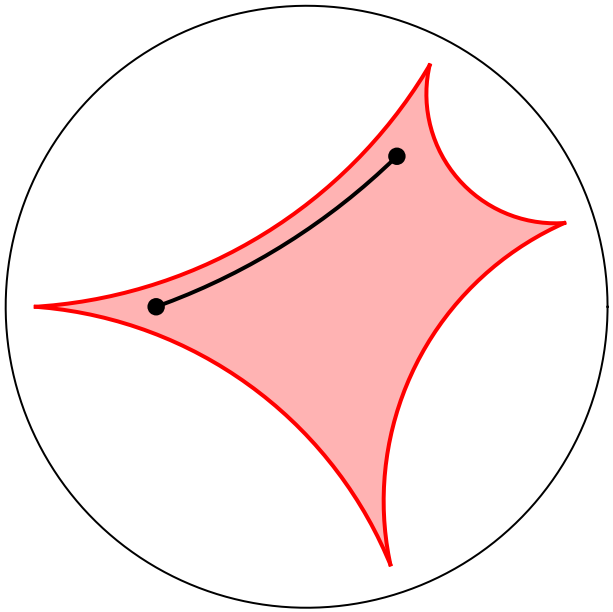
Cornell University

Penn State Geometry Seminar

Defining convexity

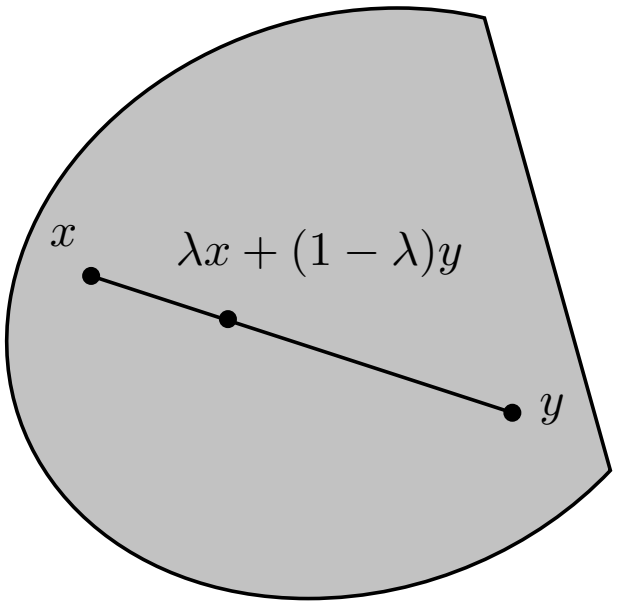
Convexity: simple geometric property with far-reaching consequences

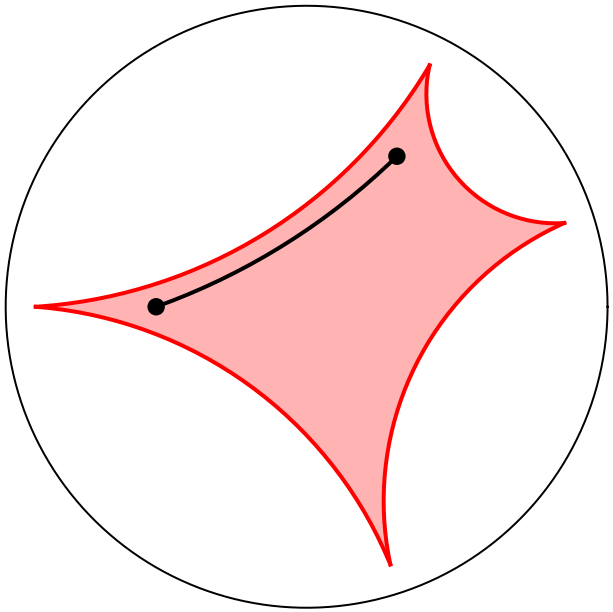




Defining convexity

Convexity: simple geometric property with far-reaching consequences





A set is **convex** if is closed under **straight line segments**, which are **shortest paths** or **geodesics** in Euclidean space.

Extension to nonlinear spaces is **natural**, but does it behave naturally?

Why such generality?

Is Euclidean convexity not useful enough?

Personal Answer: Convexity helps us solve (optimization) problems

Geodesics + **Nonpositive Curvature** cast light on Theory & Applications

Why such generality?

Is Euclidean convexity not useful enough?

Personal Answer: Convexity helps us solve (optimization) problems

Geodesics + **Nonpositive Curvature** cast light on Theory & Applications

How do **convex hulls** behave in geodesic spaces?

Why such generality?

Is Euclidean convexity not useful enough?

Personal Answer: Convexity helps us solve (optimization) problems

Geodesics + **Nonpositive Curvature** cast light on Theory & Applications

How do **convex hulls** behave in geodesic spaces?

Answer: Very strangely — today's talk

Theorem (Discovered by AI): The closed convex hull of 3 points in a complete CAT(0) space (Hadamard space) can be noncompact.

Nonpositive curvature: CAT(0) spaces

$\gamma\colon [a,b] \to X$ is a **geodesic** if $d(\gamma(t),\gamma(t')) = |t-t'|$ $(t,t' \in [a,b])$

Nonpositive curvature: CAT(0) spaces

$\gamma\colon [a,b] \to X$ is a **geodesic** if $d(\gamma(t),\gamma(t')) = |t-t'|$ $(t,t' \in [a,b])$

Hadamard Space: A **complete** **geodesic space** (X,d)

Nonpositive curvature: CAT(0) spaces

$\gamma\colon [a,b] \to X$ is a **geodesic** if $d(\gamma(t),\gamma(t')) = |t-t'|$ $(t,t' \in [a,b])$

Hadamard Space: A **complete** **geodesic space** (X,d) that is **CAT(0)**,

Nonpositive curvature: CAT(0) spaces

$\gamma\colon [a,b] \to X$ is a **geodesic** if $d(\gamma(t),\gamma(t')) = |t-t'|$ $(t,t' \in [a,b])$

Hadamard Space: A **complete** **geodesic space** (X,d) that is **CAT(0)**, meaning $d(\cdot,y)^2$ is strongly convex for all $y \in X$, i.e.

$$t \mapsto d(\gamma(t),y)^2 - t^2 \text{ is convex for any geodesic } \gamma$$

Nonpositive curvature: CAT(0) spaces

$\gamma\colon [a,b] \to X$ is a **geodesic** if $d(\gamma(t),\gamma(t')) = |t-t'|$ $(t,t' \in [a,b])$

Hadamard Space: A **complete** **geodesic space** (X,d) that is **CAT(0)**, meaning $d(\cdot,y)^2$ is strongly convex for all $y \in X$, i.e.

$$t \mapsto d(\gamma(t),y)^2 - t^2 \text{ is convex for any geodesic } \gamma$$

- Examples: Euclidean and Hilbert spaces, Hadamard manifolds

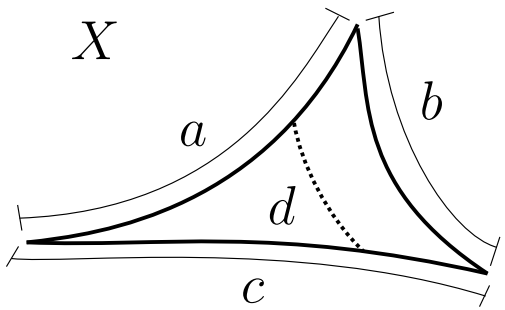
Nonpositive curvature: CAT(0) spaces

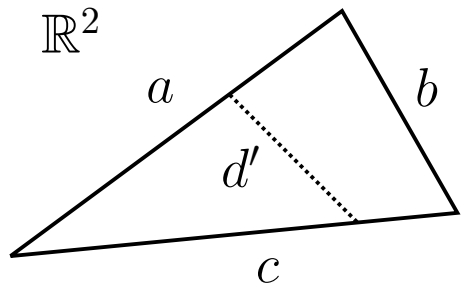
$\gamma\colon [a,b] \to X$ is a **geodesic** if $d(\gamma(t),\gamma(t')) = |t-t'|$ $(t,t' \in [a,b])$

Hadamard Space: A **complete** **geodesic space** (X,d) that is **CAT(0)**, meaning $d(\cdot,y)^2$ is strongly convex for all $y \in X$, i.e.

$$t \mapsto d(\gamma(t),y)^2 - t^2 \text{ is convex for any geodesic } \gamma$$

- Examples: Euclidean and Hilbert spaces, Hadamard manifolds
- Triangles in CAT(0) spaces are (at least as) **thin** as in Euclidean space

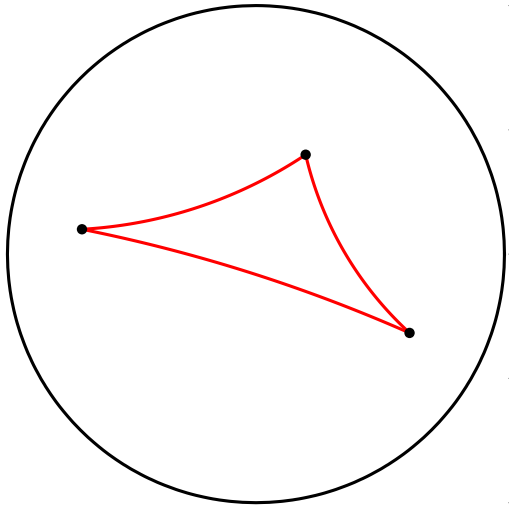




Hadamard manifolds

Hyperbolic space: open ball with metric

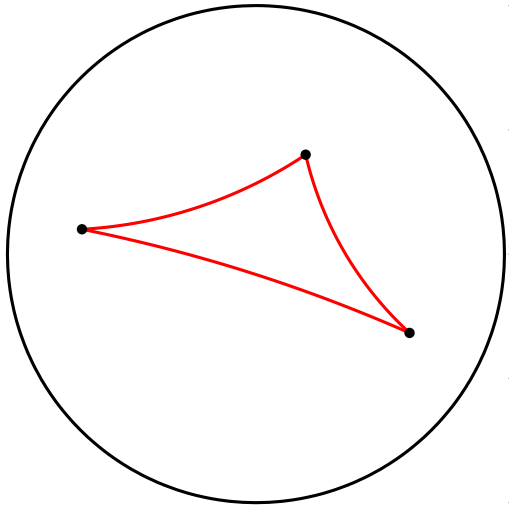
$$d(x,y) = \cosh^{-1}\left(1 + \frac{2\|x-y\|^2}{(1-\|x\|^2)(1-\|y\|^2)}\right)$$



Hadamard manifolds

Hyperbolic space: open ball with metric

$$d(x,y) = \cosh^{-1}\left(1 + \frac{2\|x-y\|^2}{(1-\|x\|^2)(1-\|y\|^2)}\right)$$

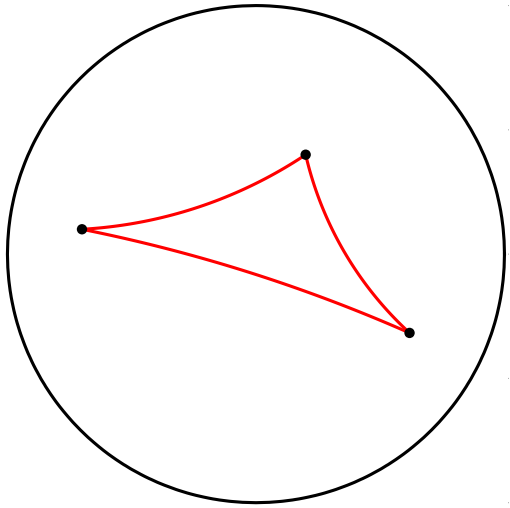


Applications: Hierarchical data embedding (De Sa. . . '18), large-margin classification (Cho. . . '18)

Hadamard manifolds

Hyperbolic space: open ball with metric

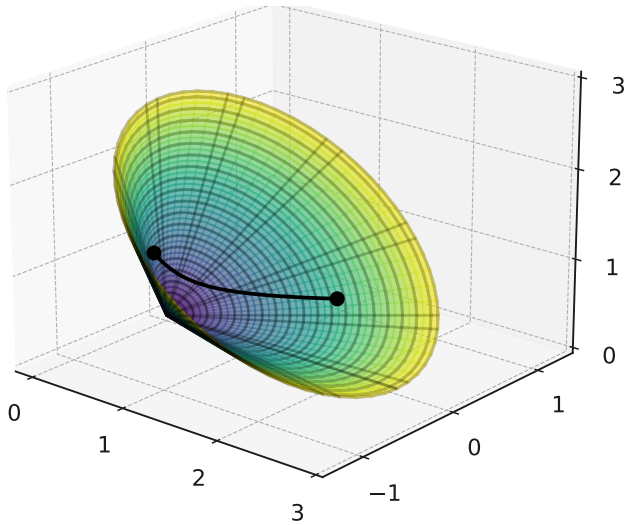
$$d(x,y) = \cosh^{-1}\left(1 + \frac{2\|x-y\|^2}{(1-\|x\|^2)(1-\|y\|^2)}\right)$$



Applications: Hierarchical data embedding (De Sa. . . '18), large-margin classification (Cho. . . '18)

Positive definite cone with metric

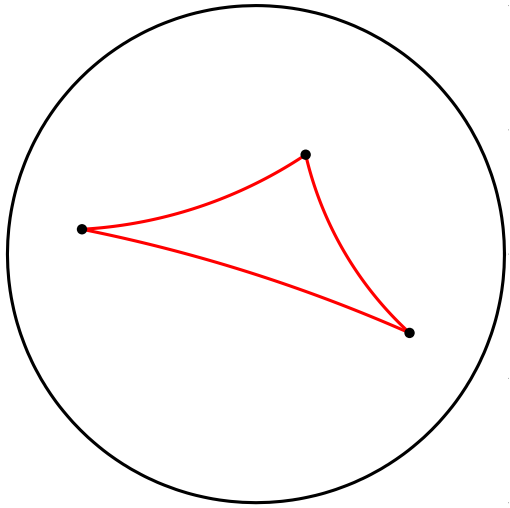
$$d(X,Y) = \|\log(X^{-\frac{1}{2}} Y X^{-\frac{1}{2}})\|_{\text{Frob}}$$



Hadamard manifolds

Hyperbolic space: open ball with metric

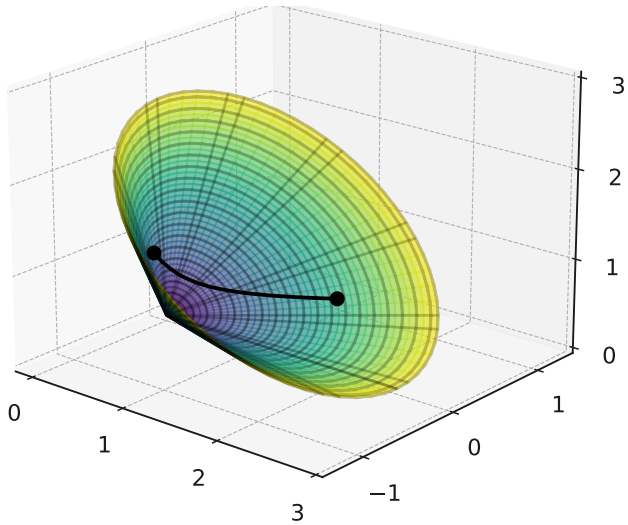
$$d(x,y) = \cosh^{-1}\left(1 + \frac{2\|x-y\|^2}{(1-\|x\|^2)(1-\|y\|^2)}\right)$$



Applications: Hierarchical data embedding (De Sa. . . '18), large-margin classification (Cho. . . '18)

Positive definite cone with metric

$$d(X,Y) = \|\log(X^{-\frac{1}{2}} Y X^{-\frac{1}{2}})\|_{\text{Frob}}$$



Applications: Covariance estimation (Auderset '05), matrix means (Bhatia. . . '12)

CAT(0) Cubical Complexes

A **complex** X of Euclidean cubes and their faces

CAT(0) Cubical Complexes

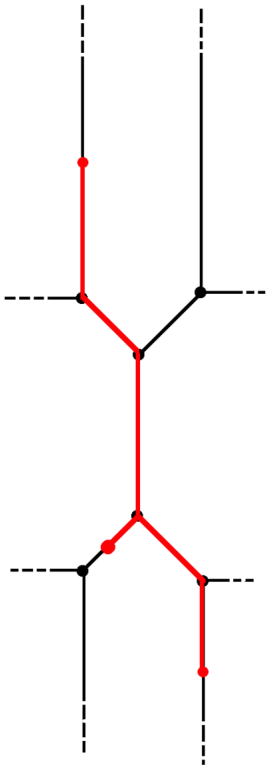
A **complex** X of Euclidean cubes and their faces, each pair of cubes sharing at most one face

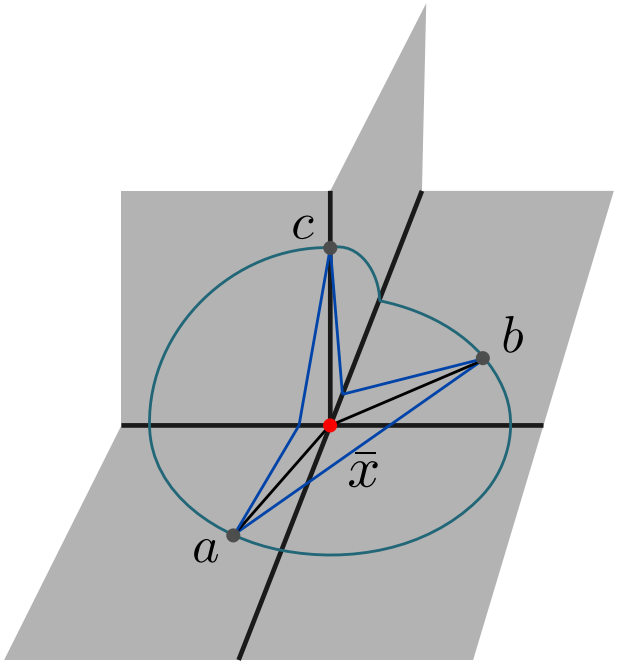
CAT(0) Cubical Complexes

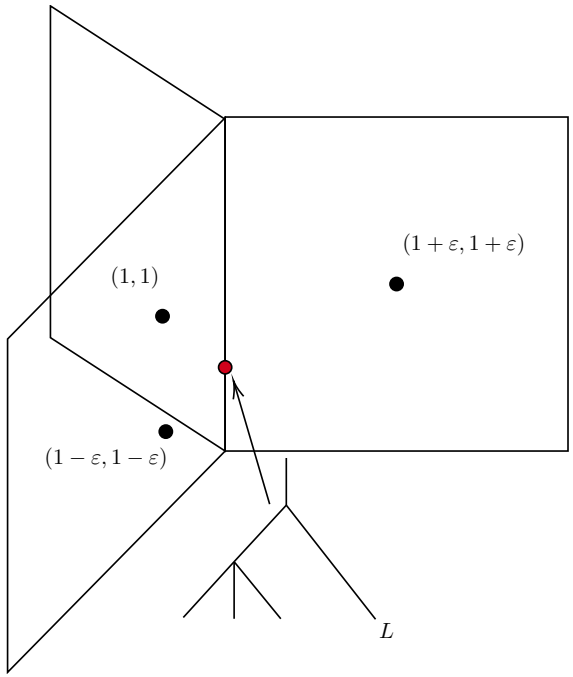
A **complex** X of Euclidean cubes and their faces, each pair of cubes sharing at most one face, is **CAT(0)** $\iff$ simply connected and **Gromov's link condition** holds

CAT(0) Cubical Complexes

A **complex** X of Euclidean cubes and their faces, each pair of cubes sharing at most one face, is **CAT(0)** $\iff$ simply connected and **Gromov's link condition** holds







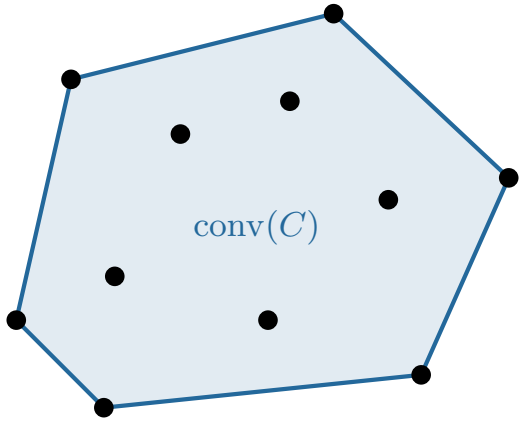
Applications: Facility location (Hansen. . . '87), phylogenetics (Billera-Holmes-Vogtmann '01), robot configurations (Ardila-Mantilla '20)

The basics of convex hulls

Convex hull of $C \subseteq X$ is $\mathrm{conv}(C)$, the **smallest convex set** containing C

$\mathbb{R}^n$: conv is an explicit and finitary operation (Carathéodory)

$$\mathrm{conv}(C) = \left\{ \sum_{i=1}^{n+1} \lambda_i x_i \mid x_i \in C, \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1 \right\}$$

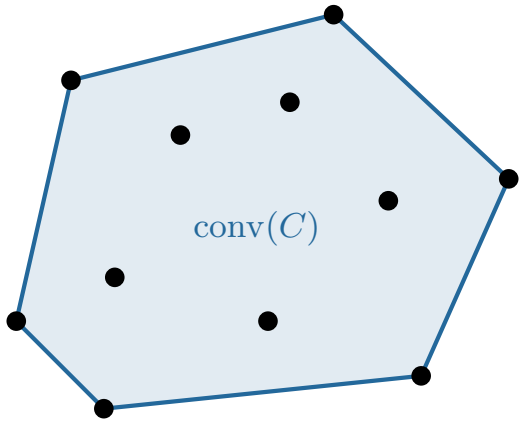


The basics of convex hulls

Convex hull of $C \subseteq X$ is $\mathrm{conv}(C)$, the **smallest convex set** containing C

$\mathbb{R}^n$: conv is an explicit and finitary operation (Carathéodory)

$$\mathrm{conv}(C) = \left\{ \sum_{i=1}^{n+1} \lambda_i x_i \mid x_i \in C, \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1 \right\}$$



Geodesic Spaces: conv is iterative with no clear finitary structure

$$\mathrm{conv}(C) = \bigcup_{n=0}^{\infty} C_n, \quad C_n = \{\text{geodesics between points in } C_{n-1}\}, \quad C_0 = C$$

Weighted means and convex hulls

Problem: **Decide** if a point x belongs to $\operatorname{conv}(\{a_1, \ldots, a_N\})$

$\mathbb{R}^n$: Solve a **linear program** $\min_\lambda \left\{ 0 \mid \lambda \in \Delta^N, \sum_{i=1}^N \lambda_i a_i = x \right\}$

Weighted means and convex hulls

Problem: **Decide** if a point x belongs to $\operatorname{conv}(\{a_1, \ldots, a_N\})$

$\mathbb{R}^n$: Solve a **linear program** $\min_\lambda \left\{ 0 \mid \lambda \in \Delta^N, \sum_{i=1}^N \lambda_i a_i = x \right\}$

Hadamard Space: Tricky — easier to **decide** if x is a **weighted mean**:

$$x \in \operatorname{mean}(A) := \text{ there is some } \lambda \in \Delta^n \text{ with } x = \operatorname{argmin} \sum_{i=1}^N \lambda_i d^2(\cdot, a_i)$$

Weighted means and convex hulls

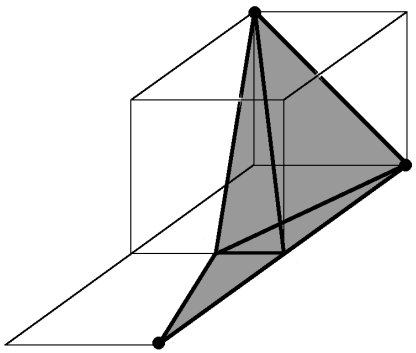
Problem: **Decide** if a point x belongs to $\operatorname{conv}(\{a_1, \ldots, a_N\})$

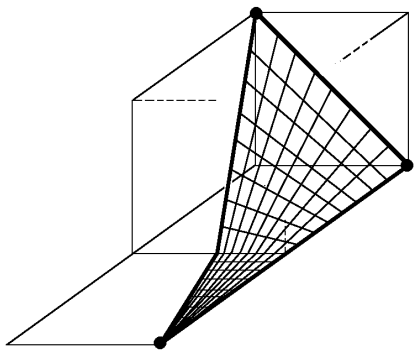
$\mathbb{R}^n$: Solve a **linear program** $\min_\lambda \left\{ 0 \mid \lambda \in \Delta^N, \sum_{i=1}^N \lambda_i a_i = x \right\}$

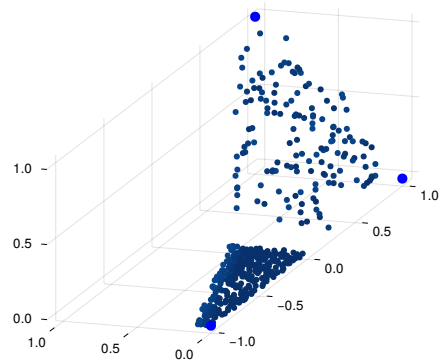
Hadamard Space: Tricky — easier to **decide** if x is a **weighted mean**:

$$x \in \operatorname{mean}(A) := \text{ there is some } \lambda \in \Delta^n \text{ with } x = \operatorname{argmin} \sum_{i=1}^N \lambda_i d^2(\cdot, a_i)$$

In general, $\operatorname{mean}(A) \subsetneq \operatorname{conv}(A)$. Computational approach in [**G**LLN, '25]:







The convex hull of 3 points can be **3-dimensional**

What else goes wrong?

Strangeness of $\mathrm{conv}(\{a_1, a_2, a_3\})$ in cube complexes exposed me to:

- **Full-dimensional** convex hulls (Lubiw. . . '20, Lytchak & Petrunin '22, Lin. . . '17, Bessenyei. . . '21)

What else goes wrong?

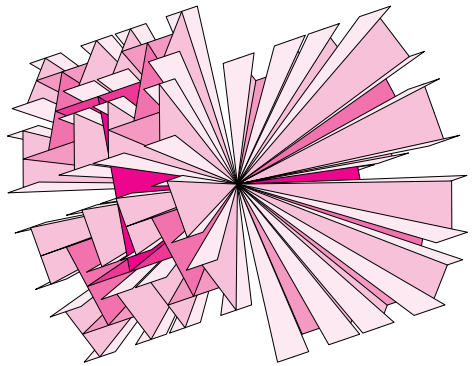
Strangeness of $\mathrm{conv}(\{a_1, a_2, a_3\})$ in cube complexes exposed me to:

- **Full-dimensional** convex hulls (Lubiw. . . '20, Lytchak & Petrunin '22, Lin. . . '17, Bessenyei. . . '21)
- Cyclic projections that do not **settle down** (Lytchak & Petrunin '22)

What else goes wrong?

Strangeness of $\mathrm{conv}(\{a_1, a_2, a_3\})$ in cube complexes exposed me to:

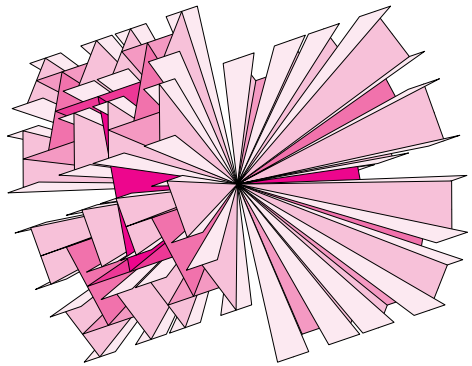
- **Full-dimensional** convex hulls (Lubiw. . . '20, Lytchak & Petrunin '22, Lin. . . '17, Bessenyei. . . '21)
- Cyclic projections that do not **settle down** (Lytchak & Petrunin '22)
- No Krein-Milman theorem: bounded Hadamard spaces without **extreme points** (Monod '16)



What else goes wrong?

Strangeness of $\mathrm{conv}(\{a_1, a_2, a_3\})$ in cube complexes exposed me to:

- **Full-dimensional** convex hulls (Lubiw. . . '20, Lytchak & Petrunin '22, Lin. . . '17, Bessenyei. . . '21)
- Cyclic projections that do not **settle down** (Lytchak & Petrunin '22)
- No Krein-Milman theorem: bounded Hadamard spaces without **extreme points** (Monod '16)



But one question, posed in [Gromov '93] and never forgotten by Petrunin:

Is the closed convex hull of a compact set compact?

Not even known for finite sets! Petrunin asked on MathOverflow in 2009

From an exercise to an open problem

Hilbert space: **finite case** is easy (general case by approximation)

Many aspects of **convexity** still work like in Hilbert spaces:

- Proximal methods and resolvents (proximal point, cyclic projections) (Bačák. . . '12, '14, '23)
- Fixed-point theory (many names e.g. Reich, Kirk, López-Acedo. . .)
- Subgradient-type algorithms (**G**LLN '26)
- Probability theory (Sturm '01, '02)

From an exercise to an open problem

Hilbert space: **finite case** is easy (general case by approximation)

Many aspects of **convexity** still work like in Hilbert spaces:

- Proximal methods and resolvents (proximal point, cyclic projections) (Bačák. . . '12, '14, '23)
- Fixed-point theory (many names e.g. Reich, Kirk, López-Acedo. . .)
- Subgradient-type algorithms (**G**LLN '26)
- Probability theory (Sturm '01, '02)

But seemingly little headway on Gromov's question for 30+ years

Mentioned frequently in work of Petrunin, Lytchak, and a survey by Bačák

So what's the answer?

Reviewing an old talk from 2024, I was reminded of the question and decided to ask AI (ChatGPT)

To my initial surprise (and skepticism), after 30 minutes it gave me:

- A Hadamard space (X, d),
- a large finite subset $A \subseteq X$,
- such that $\mathrm{conv}(A)$ is not totally bounded

So what's the answer?

Reviewing an old talk from 2024, I was reminded of the question and decided to ask AI (ChatGPT)

To my initial surprise (and skepticism), after 30 minutes it gave me:

- A Hadamard space (X, d),
- a large finite subset $A \subseteq X$,
- such that $\mathrm{conv}(A)$ is not totally bounded

Required further work to verify and simplify the construction

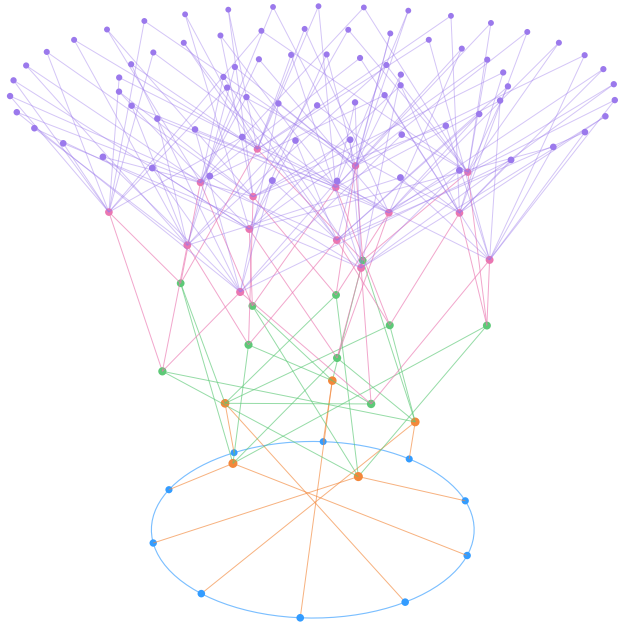
Did not resolve the case of 3 points but quickly simplified to a 3 point example (in parallel with Lytchak)

Ingredients of the construction

Construction uses standard tools from metric geometry and graph theory:

- Sequence of **metric graphs** with **long cycles** (curv. ≤ 1)

$$G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots$$

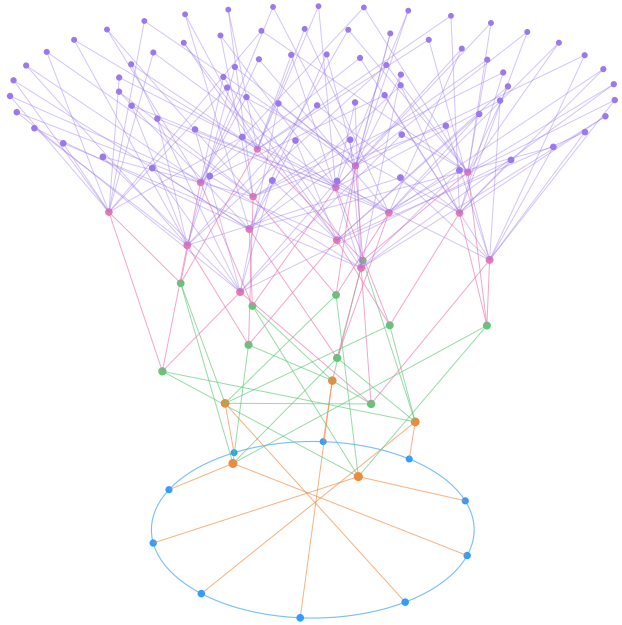


Ingredients of the construction

Construction uses standard tools from metric geometry and graph theory:

- Sequence of **metric graphs** with **long cycles** (curv. ≤ 1)

$$G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots$$



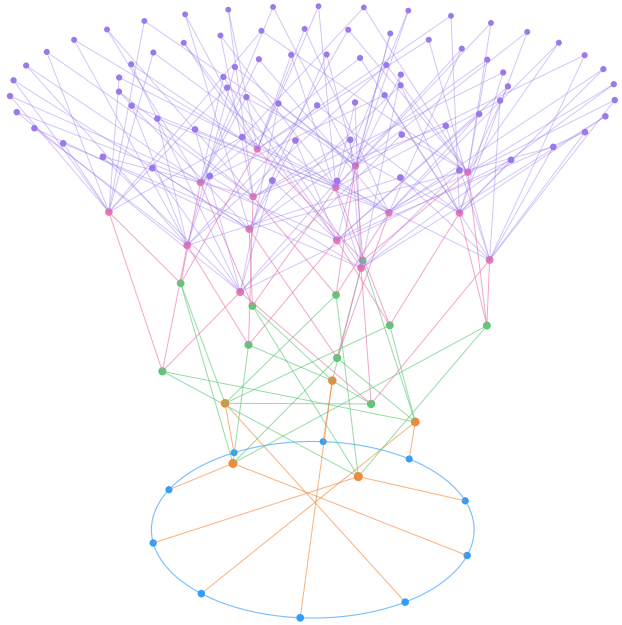
- **Edge lengths** $\{\alpha_n\}$ decay slowly ($\ell^2 \setminus \ell^1$)

Ingredients of the construction

Construction uses standard tools from metric geometry and graph theory:

- Sequence of **metric graphs** with **long cycles** (curv. ≤ 1)

$$G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots$$



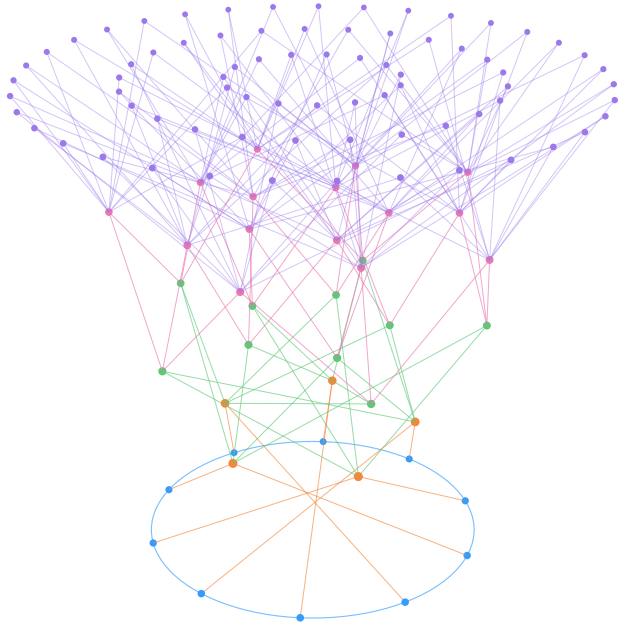
- **Edge lengths** $\{\alpha_n\}$ decay slowly $(\ell^2 \setminus \ell^1)$
- $Y = \bigcup_{i=0}^{\infty} G_i$ is **CAT(1)** $\implies$ the **cone** $C_0 Y$ is **CAT(0)**

Ingredients of the construction

Construction uses standard tools from metric geometry and graph theory:

- Sequence of **metric graphs** with **long cycles** (curv. ≤ 1)

$$G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots$$



- **Edge lengths** $\{\alpha_n\}$ decay slowly $(\ell^2 \setminus \ell^1)$
- $Y = \bigcup_{i=0}^{\infty} G_i$ is **CAT(1)** $\implies$ the **cone** $C_0 Y$ is **CAT(0)**
- **Parent-child** structure in Y induces **midpoint structure** in $C_0 Y$, enabling easy computation of an unwieldy convex hull

Basso-Krifka-Soultanis's example

The nearest relative in the literature due to Basso-Krifka-Soultanis '23:

- Space with a **conical geodesic bicombing** (weaker than CAT(0))

$$d(\sigma_{xy}(t), \sigma_{x'y'}(t)) \leq (1-t)d(x,x') + td(y,y')$$

Basso-Krifka-Soultanis's example

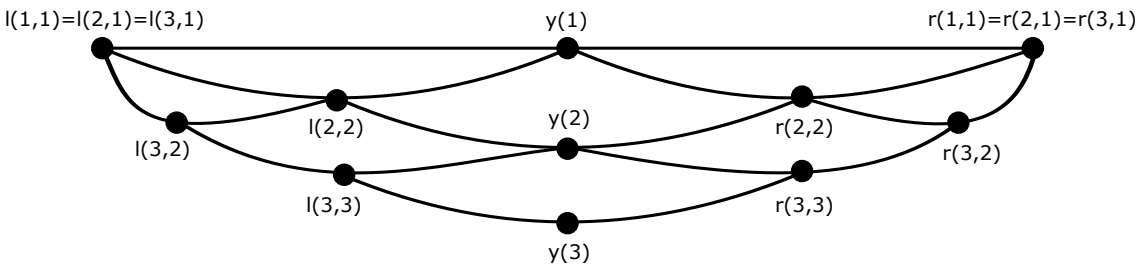
The nearest relative in the literature due to Basso-Krifka-Soultanis '23:

- Space with a **conical geodesic bicombing** (weaker than CAT(0))

$$d(\sigma_{xy}(t), \sigma_{x'y'}(t)) \leq (1-t)d(x,x') + td(y,y')$$

- Structurally similar:

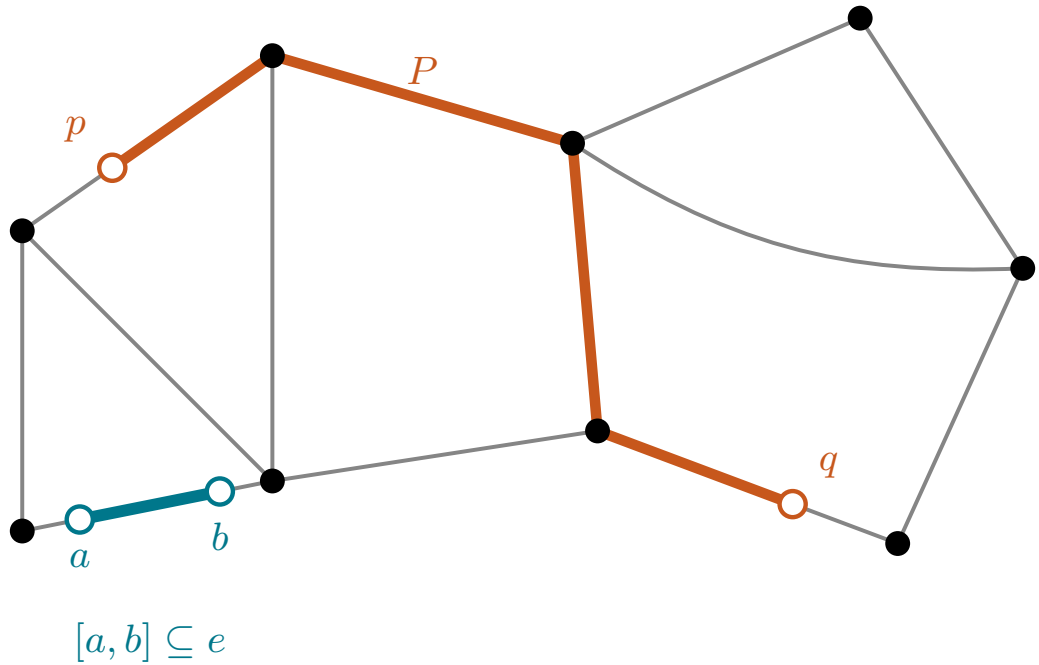
  Inductive Sequence $V_0 \to V_1 \to \cdots$ defined by taking "**midpoints**"



- **Distances** chosen to make the hull $\mathrm{conv}(V_0)$ not totally bounded
- Space is constructed to be **NPC** (no cone required)

Metric graphs

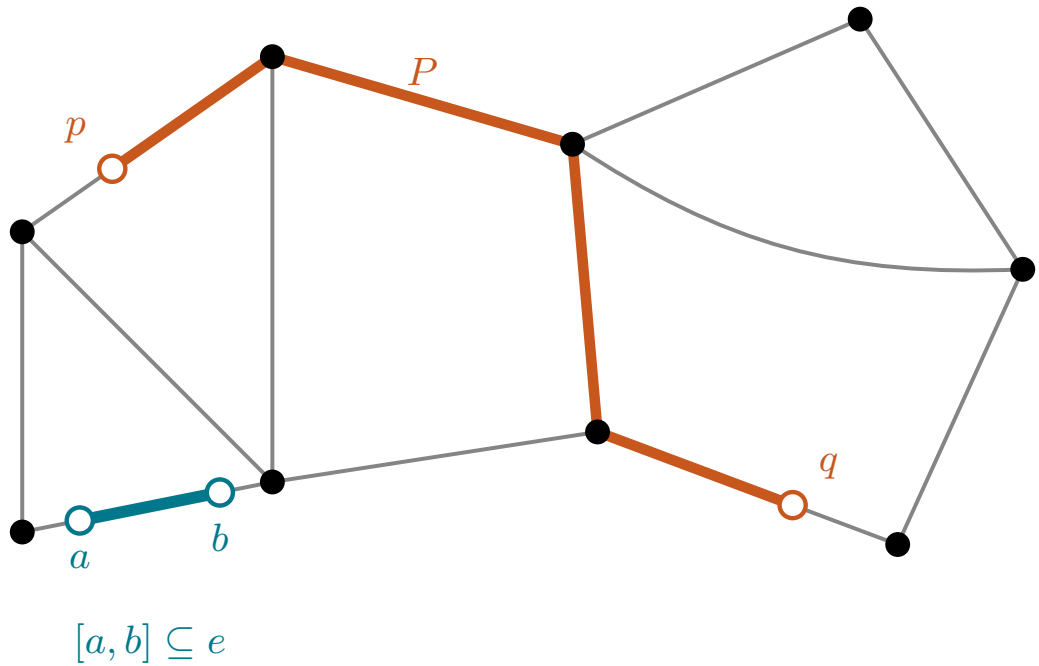
Metric graphs are graphs with positive real **edge lengths**:



Metric is length of **shortest path** built from **edge segments**

Metric graphs

Metric graphs are graphs with positive real **edge lengths**:



Metric is length of **shortest path** built from **edge segments**

Graph is **CAT(1)** $\iff$ locally CAT(1) and all cycles longer than 2π

Infinite metric graphs can be thorny but ours is easy to work with

The initial metric graph

The initial graph G_0 has two properties:

- Sufficiently many vertices M (e.g. $M = 200$)
- All cycles are length $\geq 2\pi$

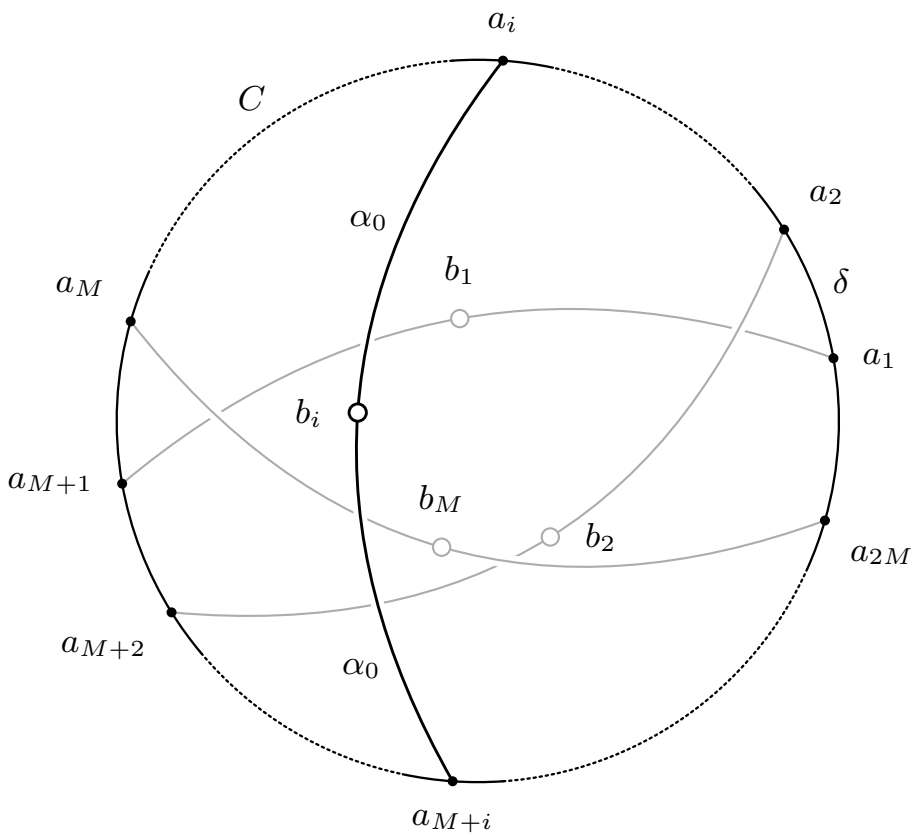
The initial metric graph

The initial graph G_0 has two properties:

- Sufficiently many vertices M (e.g. $M = 200$)
- All cycles are length $\geq 2\pi$

Explicitly:

- Cycle C of length $5\pi/2$
- $2M$ vertices V_0 around C
- V_0 are 1st generation of **parents**
- Add M **children** V_1 of opposing parents in V_0 (edge lengths α_0)

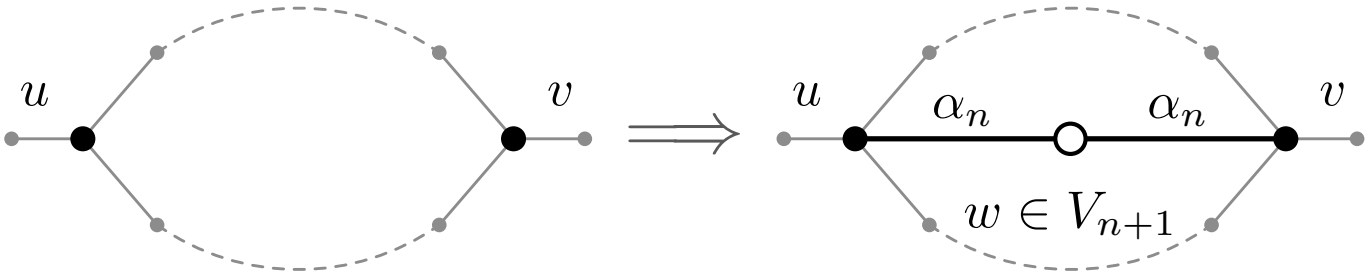


Parents and children (lesser known Russian novel)

By generation $n \geq 1$ you have a metric graph (G_n, d_n) with child set V_n

Keep the construction going while preserving two ideals:

- **Never** allow cycles shorter than 2π
- **Always** add children, but not **too many**

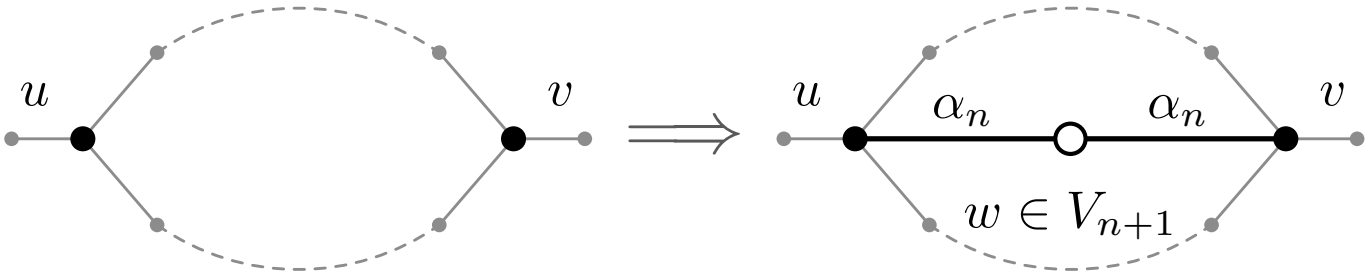


Parents and children (lesser known Russian novel)

By generation $n \geq 1$ you have a metric graph (G_n, d_n) with child set V_n

Keep the construction going while preserving two ideals:

- **Never** allow cycles shorter than 2π
- **Always** add children, but not **too many**



Augmentation Rule: If vertices $u, v \in V_n$ currently:

- are at least distance $2\pi - 2\alpha_n$ apart (in G_n + new edges)
- each have fewer than 4 children

Then add a child w via edges of length α_n (will never create short cycles)

Exponentially many children

We will need an infinite set that is not **totally bounded**:

$\Longrightarrow$ vital that sequence $\{G_n\}$ never stops growing

Exponentially many children

We will need an infinite set that is not **totally bounded**:

$\Longrightarrow$ vital that sequence $\{G_n\}$ never stops growing

Only need $|V_n| > 0$ for all $n \geq 0$ but the proof shows:

$$|V_n| \geq 50 \cdot 2^{n-2} \text{ for all } n \geq 0$$

Exponentially many children

We will need an infinite set that is not **totally bounded**:

$\Longrightarrow$ vital that sequence $\{G_n\}$ never stops growing

Only need $|V_n| > 0$ for all $n \geq 0$ but the proof shows:

$$|V_n| \geq 50 \cdot 2^{n-2} \text{ for all } n \geq 0$$

Sketch: Study vertices U_n with < 4 children

Handshaking: $|V_{n+1}| \geq 2(|V_n| - |U_n|)$

Exponentially many children

We will need an infinite set that is not **totally bounded**:

$\Longrightarrow$ vital that sequence $\{G_n\}$ never stops growing

Only need $|V_n| > 0$ for all $n \geq 0$ but the proof shows:

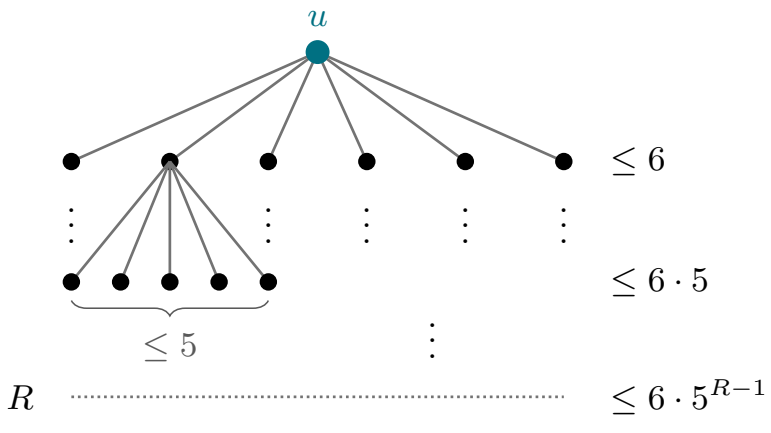
$$|V_n| \geq 50 \cdot 2^{n-2} \text{ for all } n \geq 0$$

Sketch: Study vertices U_n with < 4 children

Handshaking: $|V_{n+1}| \geq 2(|V_n| - |U_n|)$

Diameter estimate: $|U_n| = O(q^n)$ for $q < 2$

☺ because
- $|V_n|$ roughly doubles
- $|U_n|$ only grows by $q < 2$



The CAT(1) metric graph

Obtain after **infinitely many generations** a metric graph

$$Y = \bigcup_{n=0}^{\infty} G_n \quad \text{with metric} \quad d(x,y) = \lim_{m\to\infty} d_m(x,y)$$

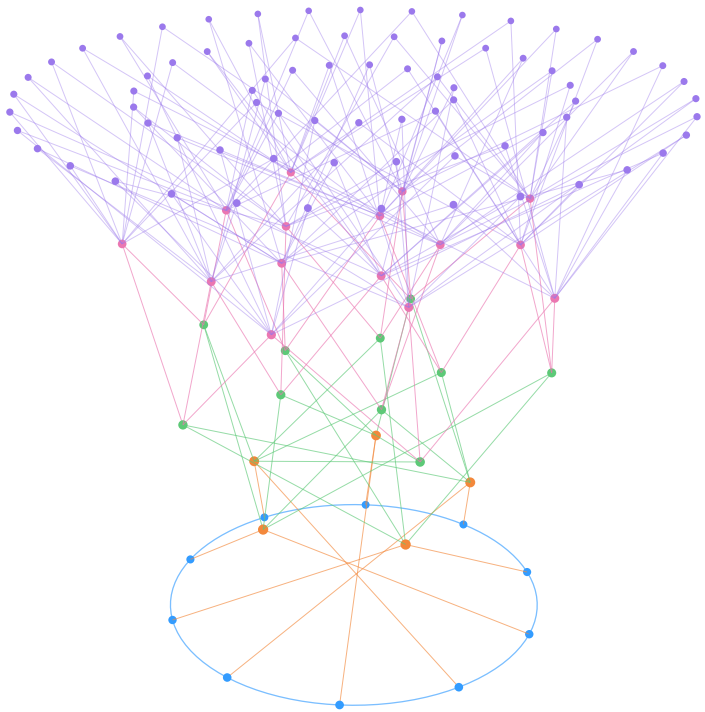
The CAT(1) metric graph

Obtain after **infinitely many generations** a metric graph

$$Y = \bigcup_{n=0}^{\infty} G_n \quad \text{with metric} \quad d(x,y) = \lim_{m\to\infty} d_m(x,y)$$

Details to check:

- d agrees with **path length metric**
- Y is a **complete, proper, geodesic space**
- Y is **CAT(1)**



From CAT(1) to CAT(0): Euclidean cones

Cone over Y is $C_0 Y = (\mathbb{R}_+ \times Y)/\sim$ where $\{0\} \times Y$ shrinks to the **tip**

with **cone metric** $d_0([t,y],[s,z]) = \sqrt{t^2 + s^2 - 2ts\cos(\min\{\pi, d(y,z)\})}$

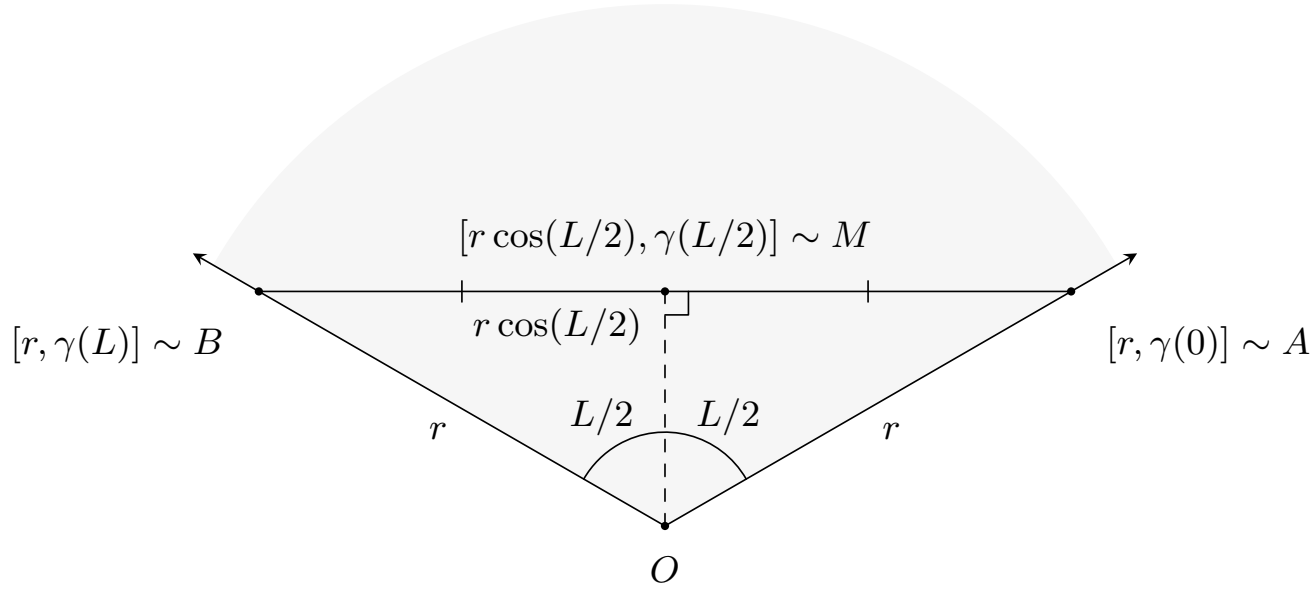
From CAT(1) to CAT(0): Euclidean cones

Cone over Y is $C_0Y = (\mathbb{R}_+ \times Y)/\sim$ where $\{0\} \times Y$ shrinks to the **tip**

with **cone metric** $d_0([t,y],[s,z]) = \sqrt{t^2+s^2-2ts\cos(\min\{\pi, d(y,z)\})}$

Key Property: C_0Y is CAT(0) $\iff$ Y is CAT(1) (**Berestovskii**)

Cone over geodesic γ **isometric** to a **Euclidean sector**:



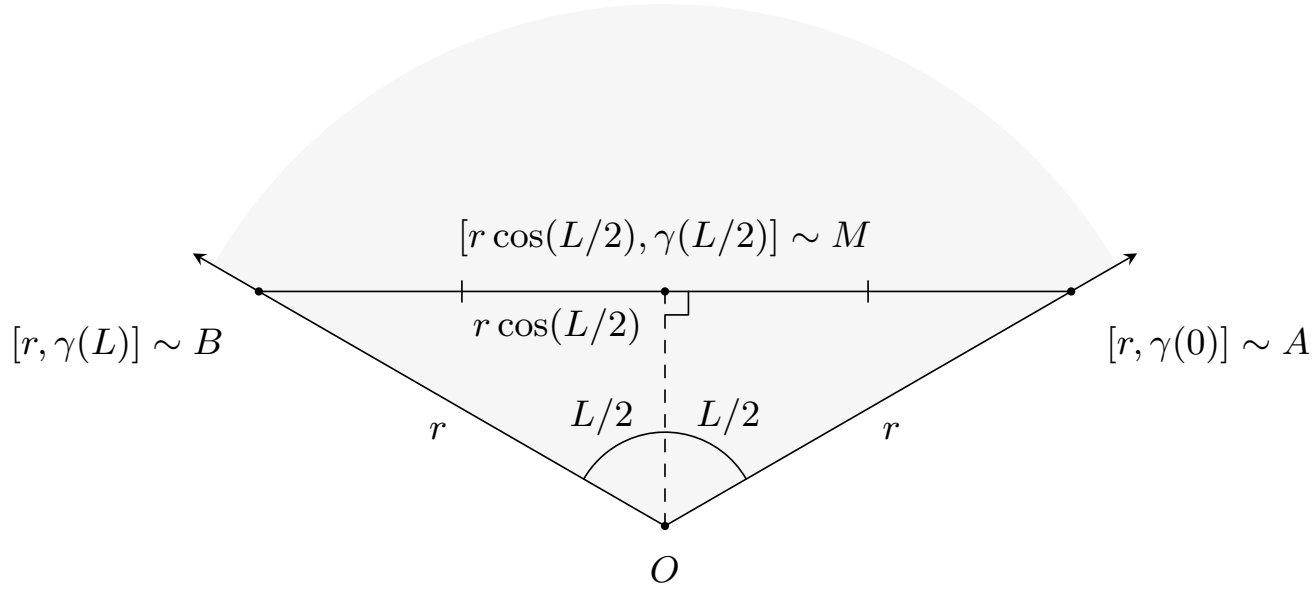
From CAT(1) to CAT(0): Euclidean cones

Cone over Y is $C_0Y = (\mathbb{R}_+ \times Y)/\sim$ where $\{0\} \times Y$ shrinks to the **tip**

with **cone metric** $d_0([t,y],[s,z]) = \sqrt{t^2+s^2-2ts\cos(\min\{\pi, d(y,z)\})}$

Key Property: C_0Y is CAT(0) $\iff$ Y is CAT(1) (**Berestovskii**)

Cone over geodesic γ **isometric** to a **Euclidean sector**:



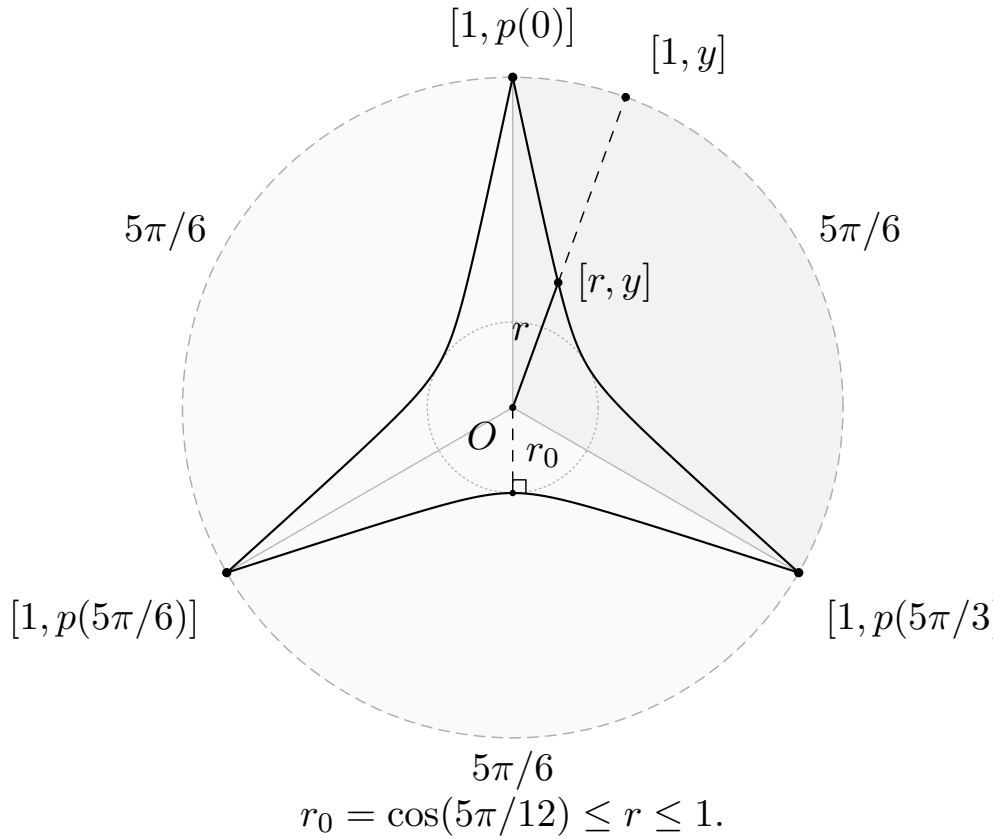
Compute geodesic midpoints in C_0Y using **elementary geometry**

N.B. Geodesic between A and B **avoids** tip O

Computing the convex hull (Step 1)

Choose $A = \{[1, a_i] \mid i = 1, 2, 3\}$ with a_i **equally spaced** around C (in G_0)

- **Geodesics** between points in A give points $[r, y]$ with some $r \geq r_0 =$ **distance from tip to geodesic**
- **Geodesics** between those points generate the tip O
- **Contract radially** to get all points $Q_0 = \{[r_0, v] \mid v \in V_0\}$



$\implies Q_0 \subseteq \mathrm{conv}(A)$.

Computing the convex hull (Step 2)

Define $Q_n = \{[r_n, w] \mid w \in V_n\}$

Want r_n such that $[r_n, w]$ is the midpoint of $[r_{n-1}, u], [r_{n-1}, v]$

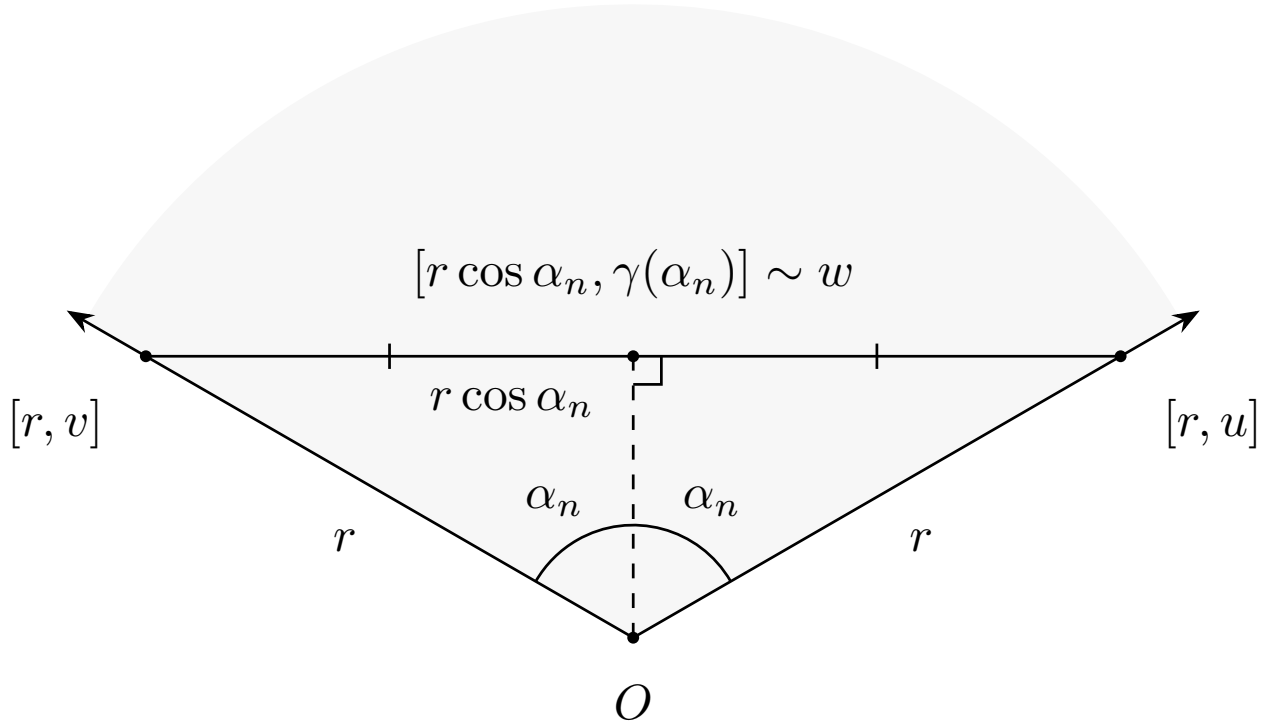
(u, v are the parents of w in V_{n-1})

Computing the convex hull (Step 2)

Define $Q_n = \{[r_n, w] \mid w \in V_n\}$

Want r_n such that $[r_n, w]$ is the midpoint of $[r_{n-1}, u], [r_{n-1}, v]$

(u, v are the parents of w in V_{n-1})



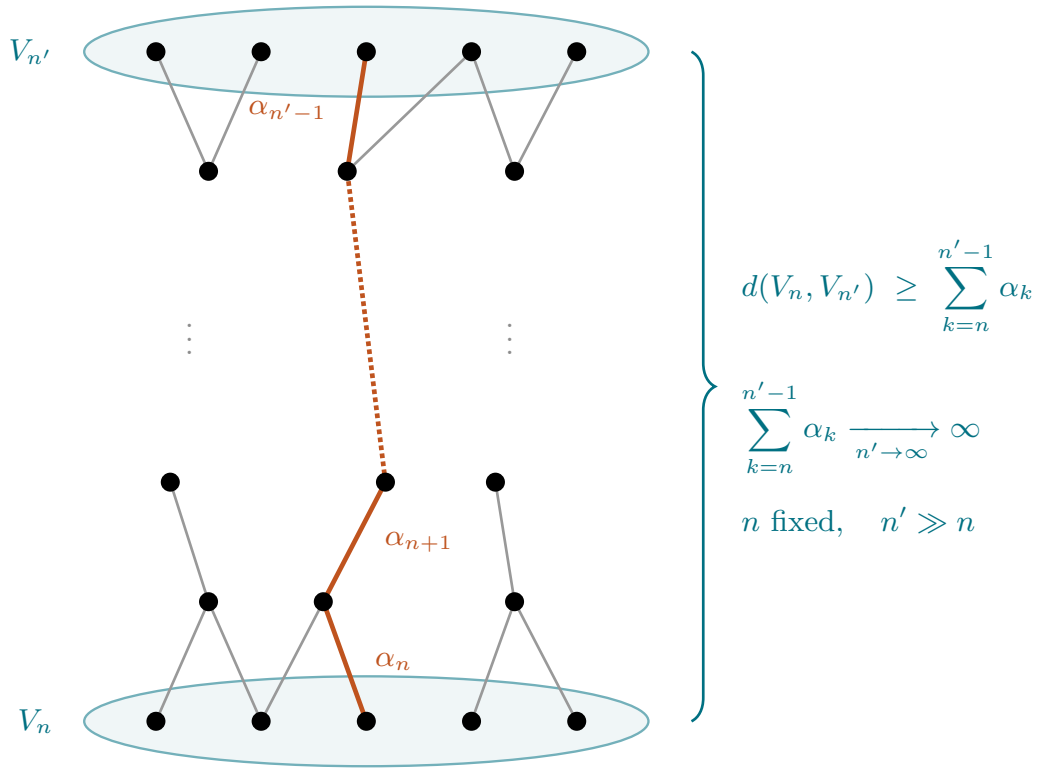
Geometry dictates our choice: $L = d(u, v) = 2\alpha_n$, so $r_n = r_{n-1}\cos(\alpha_n)$

By induction, $Q_n \subseteq \text{conv}(A)$ for all $n \geq 0$

A well-separated subsequence

Need to use properties of the edge lengths $\{\alpha_n\} \in \ell^2 \setminus \ell^1$:

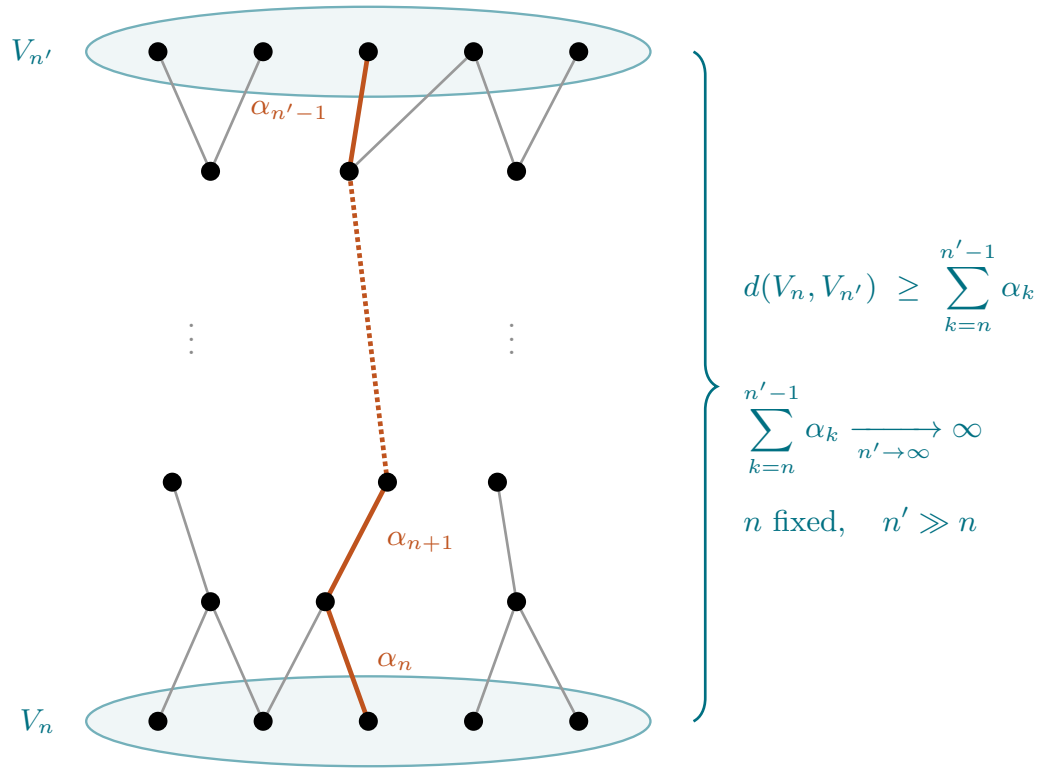
Choose a subsequence $x_j = [r_{n_j}, v_{n_j}] \in Q_{n_j}$ such that $V_{n_{j+1}}$ is separated from V_{n_j} by distance π (uses $\{\alpha_n\} \notin \ell^1$)



A well-separated subsequence

Need to use properties of the edge lengths $\{\alpha_n\} \in \ell^2 \setminus \ell^1$:

Choose a subsequence $x_j = [r_{n_j}, v_{n_j}] \in Q_{n_j}$ such that $V_{n_{j+1}}$ is separated from V_{n_j} by distance π (uses $\{\alpha_n\} \notin \ell^1$)



$$
\begin{aligned}
d_0(x_i, x_j)^2 &= r_{n_i}^2 + r_{n_j}^2 - 2 r_{n_i} r_{n_j} \cos(\min\{\pi, d(v_{n_i}, v_{n_j})\}) \\
&= (r_{n_i} + r_{n_j})^2 && \text{(separation of } V_{n_i}, V_{n_j}\text{)} \\
&\geq 4 \inf_i r_{n_i}^2 > 0 && \text{(square-summability of } \{\alpha_n\}\text{)}
\end{aligned}
$$

Summary

We've shown that

- conv(A) contains $Q = \bigcup_n Q_n$,
- and that Q is not totally bounded.

Q: What stronger assumptions besides CAT(0) must be added to make convex hulls well-behaved?

A. Goodwin. A finite set with noncompact closed convex hull in a Hadamard space. arXiv:2609.06455v2 (new version today!)

A. Lytchak. Non-compact convex hulls in CAT(0) spaces revisited. arXiv:2609.12189