Convex Optimization in Geodesic Spaces: Problems and Algorithms

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Joint work with A.S. Lewis, G. López-Acedo, and A. Nicolae

Young Researchers Workshop 2026

Familiar tools for new problems

What geometric structure enables convex optimization?

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Emergent Answer: Geodesics and Nonpositive Curvature

This talk

Fundamentals of convex optimization in **geodesic metric spaces:**

- Convexity
- Subgradients
- Applications: Weighted Means, Minimum Enclosing Balls

Geodesics and nonpositive curvature

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$t \mapsto d(\gamma(t),y)^2 - t^2$ is convex for any geodesic γ

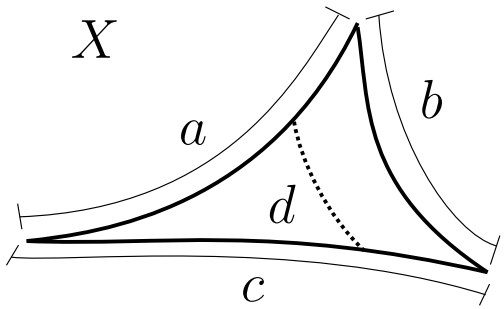
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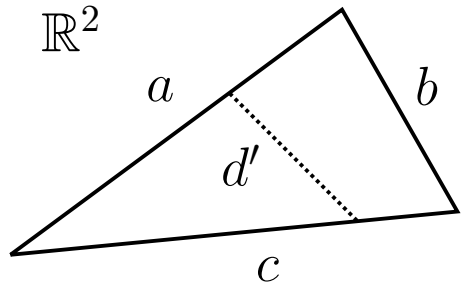
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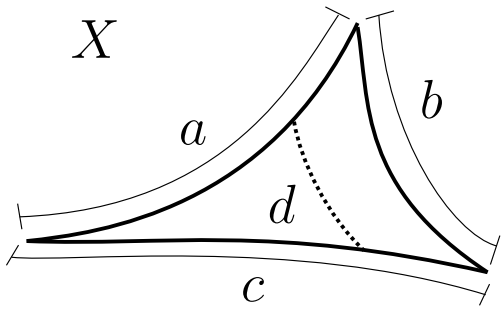
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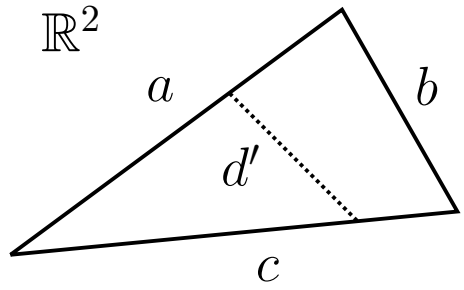
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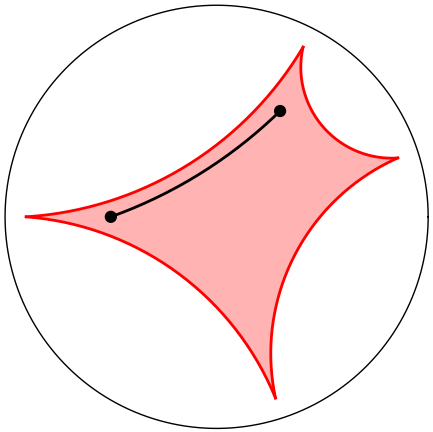


- Triangles are (at least as) **thin** as in Euclidean space
- $f\colon X \to \mathbb{R}$ is **convex** $\equiv f \circ \gamma$ is convex for any geodesic γ
- Easy Examples: Euclidean and Hilbert spaces

Hadamard spaces: smooth and nonsmooth

Smooth: Hadamard manifolds

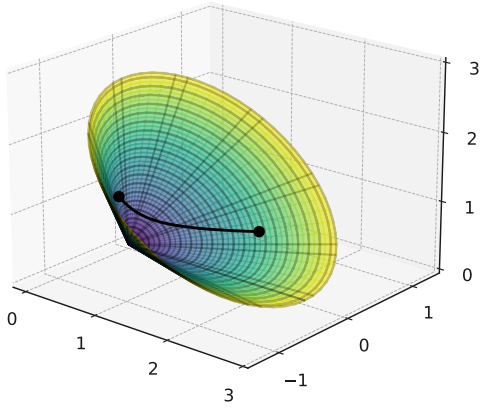
Hyperbolic space



Hierarchical data embedding (De Sa... '18)

Large-margin classification (Cho... '18)

Positive definite cone



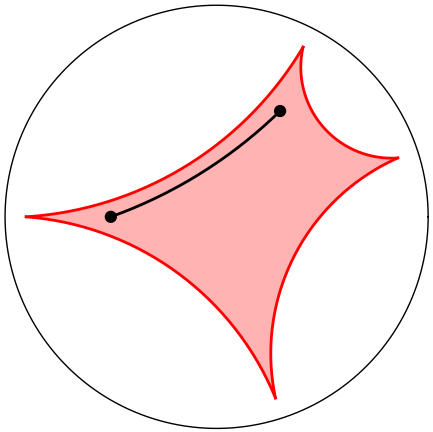
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Matrix means (Bhatia... '12)

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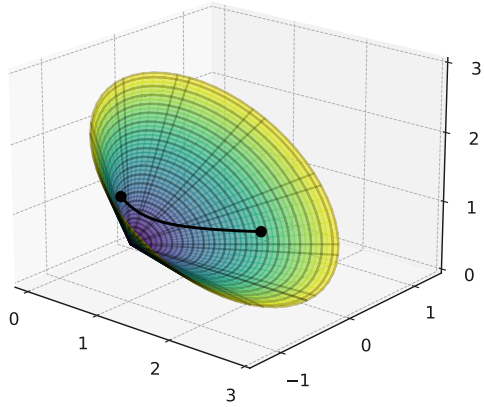
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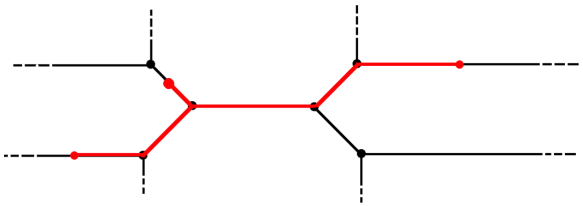


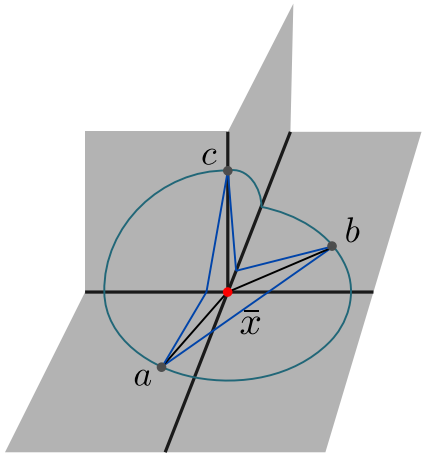
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Nonsmooth: cubical complexes

Curvature ≤ 0 $\iff$ simply connected + **Gromov's link condition**





Facility location
(Handler '73, Hansen... '87)

Phylogenetics
(Billera–Holmes–Vogtmann '01)

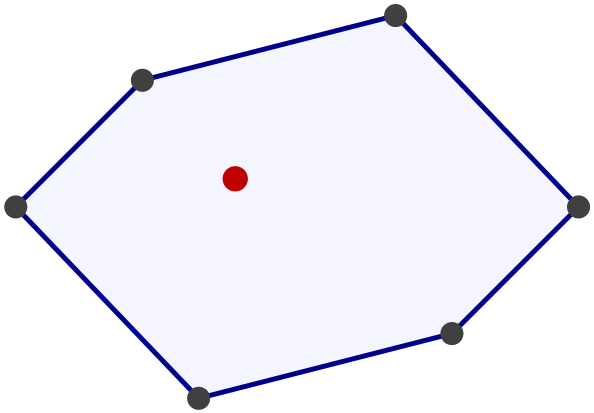
Robot configurations
(Ardila-Mantilla '20)

Fundamental optimization problems

Given data $a_1, \ldots, a_k \in X$, we might ask:

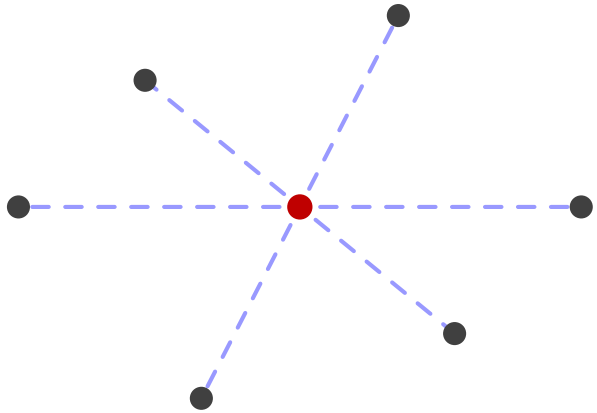
Convex-hull membership

Is $\bar{x} \in \operatorname{conv}\{a_i\}$?



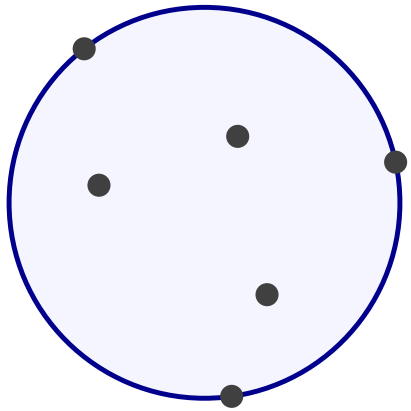
Compute a p-mean

Minimize $\sum_i w_i d^p(x, a_i)$



Minimum enclosing ball

Minimize $\max_i d(x, a_i)$



What do we know about convexity?

"Is $\overline{\text{conv}}\{a_1, a_2, a_3\}$ compact?" – Gromov, 1993...

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Instead, **decide computationally** if x is a **weighted mean**: (**G**LLN '25)

$$x \in \text{mean}(A) := \text{ there is some } w \in \Delta^k \text{ with } x = \text{argmin} \sum_{i=1}^{k} w_i d^2(\cdot, a_i)$$

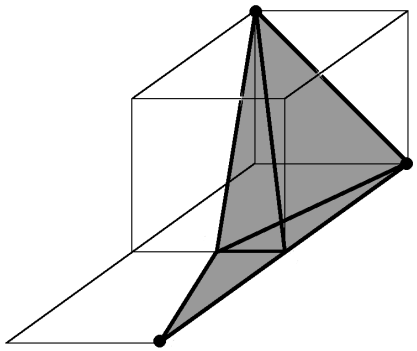
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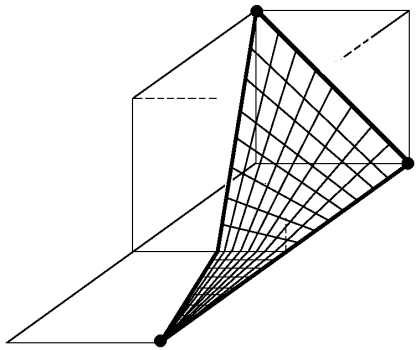
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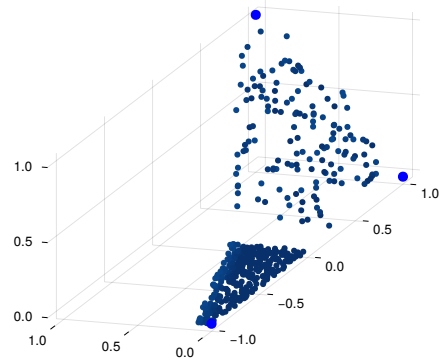
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Means $\subsetneq$ Hulls: the convex hull of 3 points can be **3-dimensional**







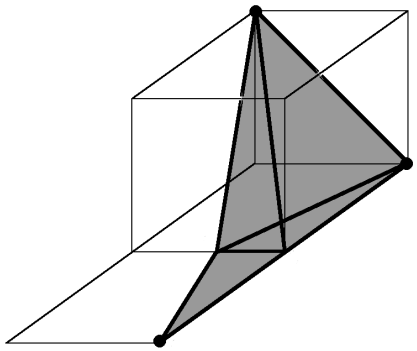
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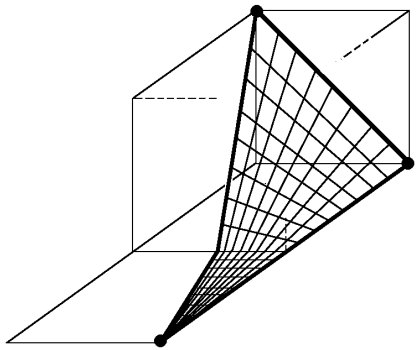
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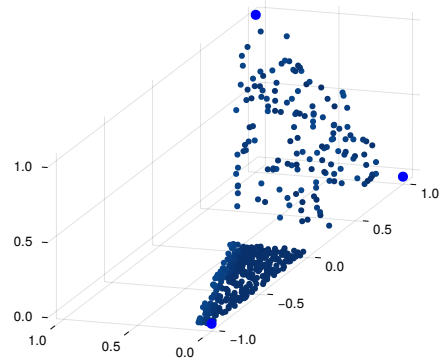
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Reply to Gromov: "$\overline{\text{conv}}\{a_1, a_2, a_3\}$ can be **noncompact**!" – AI (**G** '26)

What is a subgradient?

In Euclidean space $\mathbb{E}$, **subgradients** are dual objects: $v \in \partial f(x)$ means

$$f(y) \geq f(x) + \langle v, y - x \rangle \text{ for all } y \in \mathbb{E}$$

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Natural extension to Hadamard manifolds but ...

- Complexity needs **lower curvature bound** (Zhang-Sra '16)
- How to define in **non-manifolds**?

Lacking linear structure, turn to metric tools: **distances and geodesics**

Busemann functions — a fresh take on convexity

Given a (geodesic) ray $r\colon \mathbb{R}_+ \to X$, the **Busemann function** $b_r\colon X \to \mathbb{R}$,

$$b_r(x) := \lim_{t\to\infty} (d(x, r(t)) - t)$$

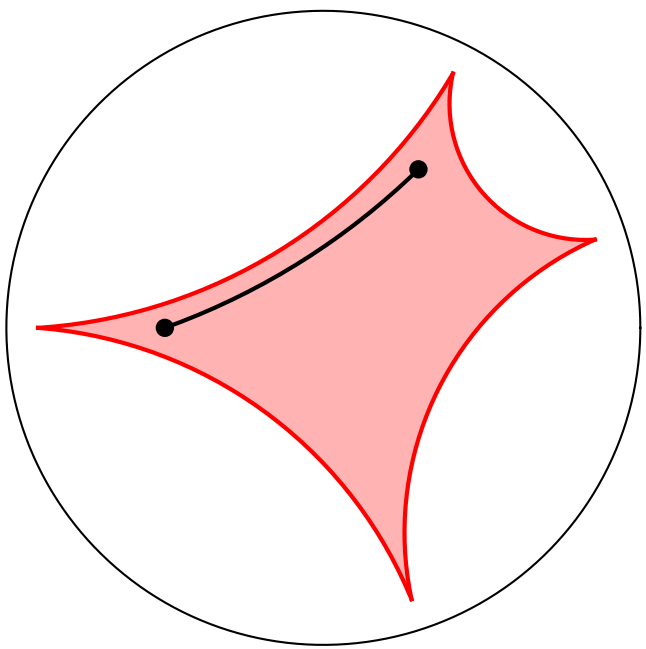
Euclidean Space: **affine functions**

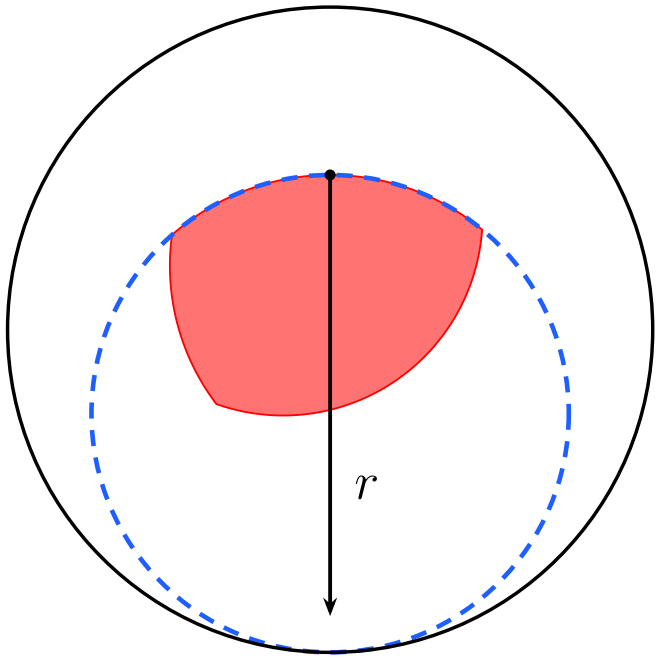
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Euclidean Space: **affine functions**





Level sets $\{b_r \leq c\}$ called **horoballs** — generalizes **halfspaces**

A new kind of subgradient

Characterization of (classical) subgradients: $v \in \partial f(x)$ iff

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Two pieces of data:

- r is a ray issuing from x (**direction**)
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$\implies$ convexity (**stronger** in general: $x \mapsto \operatorname{dist}_C(x)$)

Other subgradients: Kakavandi-Amini '10, LLN '23, Criscitiello-Kim '25

Are there any Busemann subdifferentiable functions?

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GLLN '26b gives examples and calculus for combining them:

Example	$\mathbf{f(x)}$	**Ray**	**Speed**
Euclidean	Convex, $v \in \partial f(x)$	$r(t) = x - t\frac{v}{\|v\|}$	$\|v\|$
p-**Distance**	$d(x,a)^p$	r from x to a	$pd(x,a)^{p-1}$
Busemann	$b_{r'}(x)$	r from x with $r \sim r'$	1
Distance to Ball	$\mathrm{dist}(x, B_\rho(a))$	r from x to a	$\mathbb{1}_{B_\rho(a)^c}(x)$
Distance to Horoball	$\mathrm{dist}(x, \{b_{r'} \leq 0\})$	r from x with $r \sim r'$	$\mathbb{1}_{\{b_{r'}>0\}}(x)$

Closed under finite max, composition with $\nearrow$ convex

But no **sum rule**!

Splitting methods for composite problems

Bertsekas-Nedic '01 propose **splitting** algorithms for $\min_C f := \sum_{i=1}^k f_i$

Euclidean Subgradient Splitting Method

Repeat for $n \geq 0$:

Choose $t_n > 0$ and $i(n) \sim \mathrm{Unif}\{1, \ldots, k\}$.

Choose $v^n \in \partial f_{i(n)}(x^n)$.

$$x^{n+1} \leftarrow \mathrm{proj}_C\left(x^n - t_n \|v^n\| \frac{v^n}{\|v^n\|}\right) \quad \left(\text{Dir. } -\frac{v^n}{\|v^n\|}, \text{ speed } \|v^n\|\right)$$

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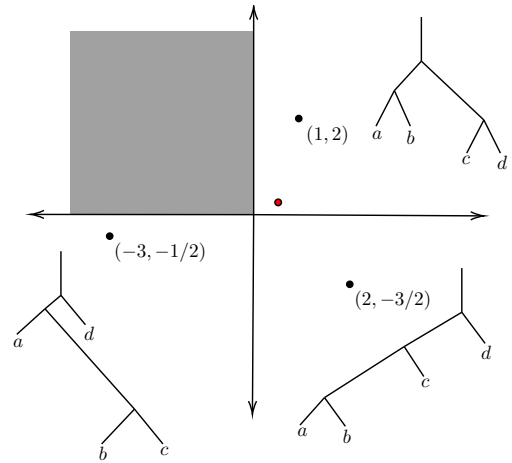
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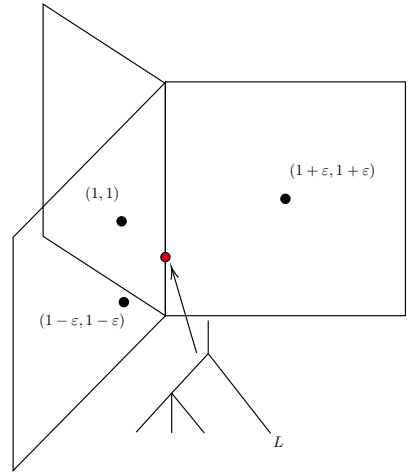
Theorem: Closed convex C, Lipschitz and **B-sd** f_i, with minimizer in C:

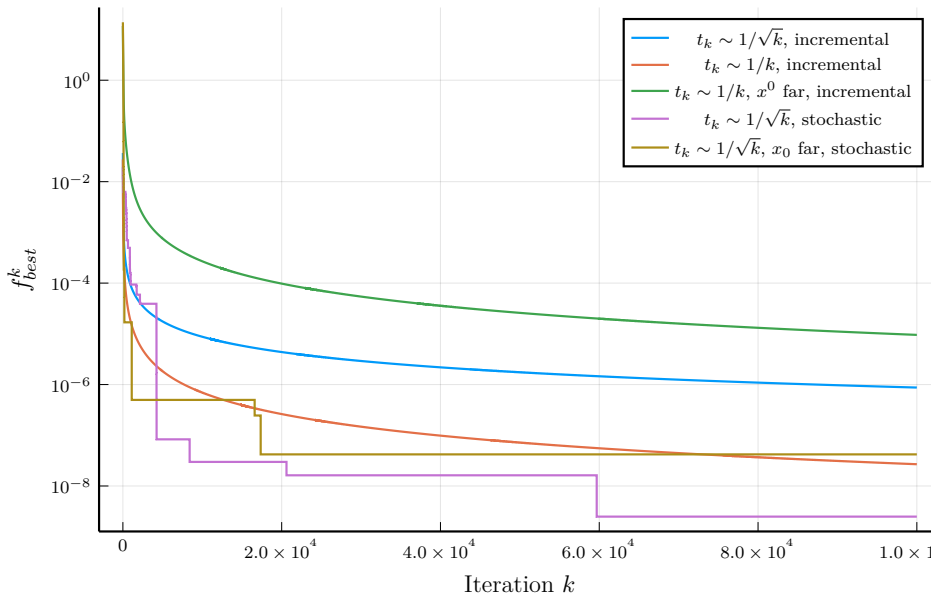
$$\mathbb{E}\left[\min_{n=1,\ldots,N} f(x^n)\right] - \min_C f = O\left(\frac{1}{\sqrt{N}}\right) \qquad \left(t_n \sim \frac{1}{\sqrt{n}}\right)$$

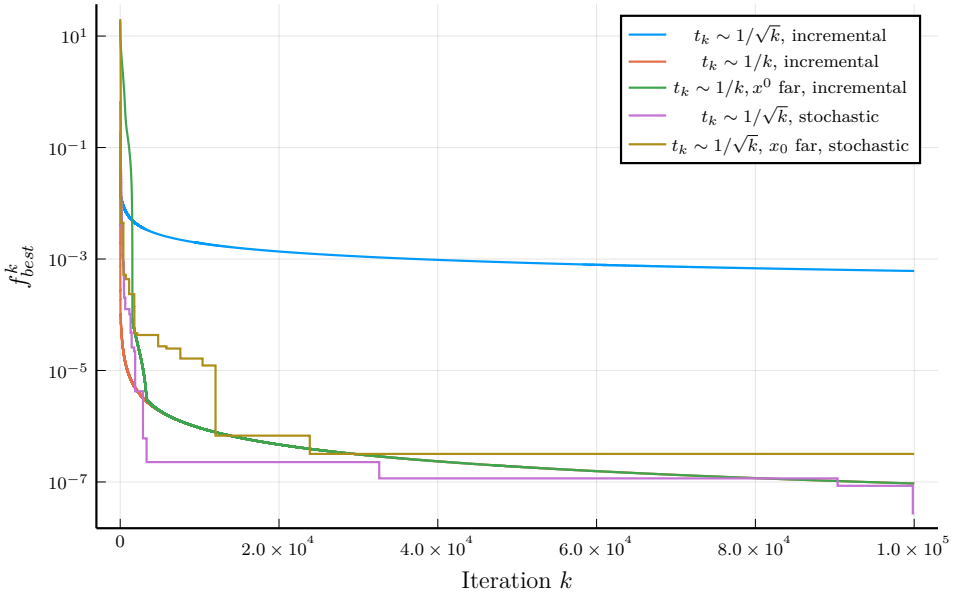
Application: 1-means in tree space

BHV Tree Space: Binary trees with nonnegative edge lengths









Application: minimum enclosing balls

$$\min_{x \in X} \max_{i=1,\dots,k} d(x, a_i)$$

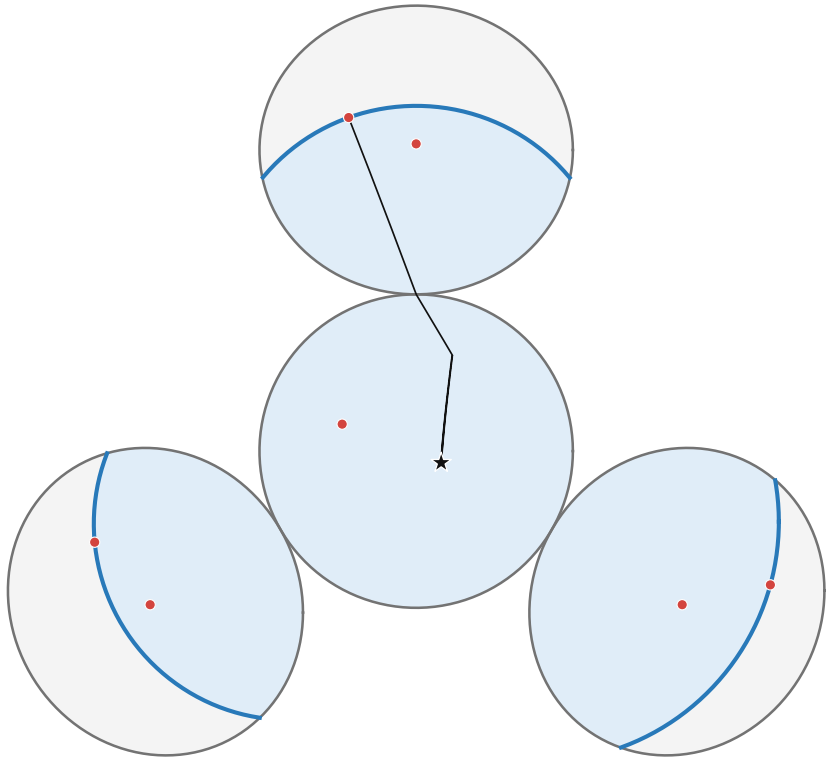
Min. Enclosing Ball Algorithm

For $k = 1, 2, \dots$

Choose $\bar{a}^k$ furthest from x^k

$$x^{k+1} = \frac{k-1}{k+1} x^k + \frac{2}{k+1} \bar{a}^k$$

End for



Example in Curvature ≤ 1

Euclidean (Bâdoiu-Clarkson '03), Riemannian (Arnaudon-Nielsen '13)

Hadamard (**G**LLN '26a), Curvature $\leq \kappa$ (**G**L '26)

Summary

Convex optimization in geodesic spaces:
many problems, a few **algorithms**

GLLN '26a: Circumcenters and mean sets in Hadamard space: horospherical subgradient methods. *Math. of OR*, 2026.

GLLN '26b: Stochastic and incremental subgradient methods for convex optimization on Hadamard spaces. *Math. Prog.*, 2026.

GLLN '25: Recognizing weighted means in geodesic spaces. *Found. of Comp. Math.*, 2025.

GL '26: Minimal enclosing balls via geodesics. arXiv:2603.15488, 2026.

G '26: A finite set with noncompact closed convex hull in a Hadamard space. arXiv:2609.06455, 2026.