These notes were prepared by Ariel Goodwin for MATH 248 at McGill University as taught by Pengfei Guan.

1 Conservative Vector Fields

Recall that a C^1 vector field $\mathbf{F}\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is conservative if there exists a C^1 function $f\colon \Omega \to \mathbb{R}$ such that $\mathbf{F}(x) = \nabla f(x)$ for all $x \in \Omega$. Conservative vector fields have many nice properties that we already know: line integrals of conservative vector fields obey a fundamental theorem of calculus (in particular, they depend only on the endpoints of the curve), and their curl is zero (whenever curl is defined, e.g. $n = 2, 3$). Note that the curl of a vector field $\mathbf{F} = (P, Q)$ on $\mathbb{R}^2$ is taken to be $Q_x - P_y$ (reminds us of Green's theorem).

This gives us some methods for checking if a vector field is not conservative (non-zero curl or two distinct values for line integrals with different curves but the same endpoints). We have an example showing that being conservative is not equivalent to zero curl:

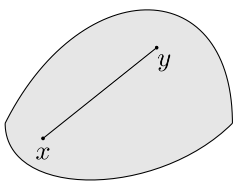
$$\mathbf{F}(x,y,z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right),\ \forall (x,y,z) \in \mathbb{R}^3 \setminus \mathbb{R}_z$$

where $\mathbb{R}_z$ is the z-axis. It would be nice if zero curl implied conservative. It turns out that for a large class of vector fields defined on certain regions that this is the case.

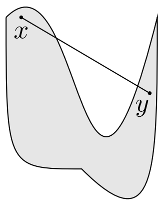
Definition 1: Convexity

A set $C \subseteq \mathbb{R}^n$ is **convex** if $\lambda x + (1-\lambda)y \in C$ for all $x, y \in C$, $\lambda \in [0,1]$.

This definition says that for any two points x, y in a convex set C, the line connecting x to y is contained in C. Convex sets are very natural, and drawing some pictures allows us to get a feel for them.



Convex set



Non-convex set

As an aside, convexity is a simple notion with many deep consequences throughout mathematics. The study of convex sets and functions forms the basis of convex analysis, a rich geometric tool that finds much use in optimization, for example.

Theorem 1: Conditions for Conservative Fields

Let $\Omega \subseteq \mathbb{R}^3$ be convex, let $W = \Omega \setminus \{p_1, \ldots, p_k\}$ where $p_1, \ldots, p_k$ is a finite collection of points in Ω and suppose $\mathbf{F} \colon W \to \mathbb{R}^3$ is C^1. Then the following are equivalent:

a) For any oriented simple closed curve $\gamma \subseteq W$, $\int_\gamma \mathbf{F} \cdot d\mathbf{s} = 0$.

b) $\int_{\gamma_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\gamma_2} \mathbf{F} \cdot d\mathbf{s}$ for any oriented simple curves $\gamma_1, \gamma_2 \subseteq W$ that share the same endpoints.

c) $\mathbf{F}$ is conservative, i.e., $\mathbf{F} = \nabla f$ for some C^1 function f on W.

d) $\operatorname{curl} \mathbf{F} = 0$

We remark that this theorem holds more generally if Ω is assumed only to be *simply connected* rather than convex, which is significantly weaker. The topological notion of simple connectivity is beyond the scope of this course, however.

Exercise 1. Determine if the following vector fields are conservative, and if so find a corresponding f.

a) $\mathbf{F}(x, y) = (\cos xy - xy \sin xy, -x^2 \sin xy)$

b) $\mathbf{F}(x, y) = (x\sqrt{x^2y^2 + 1}, y\sqrt{x^2y^2 + 1})$

c) $\mathbf{F}(x, y) = (2x \cos y + \cos y, -x^2 \sin y - x \sin y)$

Exercise 2. Let $\mathbf{F}(x, y, z) = (2xyz + \sin x, x^2z, x^2y)$. Find a function f such that $\mathbf{F} = \nabla f$.

Exercise 3. Suppose $\mathbf{F} = (F_1, F_2, F_3)$ where $F_i(tx, ty, tz) = tF_i(x, y, z)$ for all $t \in \mathbb{R}, (x, y, z) \in \mathbb{R}^3$. Suppose also that $\operatorname{curl} \mathbf{F} = 0$. Prove that $\mathbf{F} = \nabla f$, where

$$f(x, y, z) = \frac{1}{2} (xF_1(x, y, z) + yF_2(x, y, z) + zF_3(x, y, z))$$

Hint: Use Assignment 2, Q4.

Exercise 4. Let $\mathbf{F}(x, y, z) = (e^x \sin y, e^x \cos y, z^2)$. Compute $\int_\gamma \mathbf{F} \cdot d\mathbf{s}$ where $\gamma(t) = (\sqrt{t}, t^3, e^{\sqrt{t}})$ for $0 \leq t \leq 1$.

Exercise 5. Prove that if $\mathbf{F} = (F_1, F_2, F_3)$ is a C^1 vector field on $\mathbb{R}^3$ with $\operatorname{div} \mathbf{F} = 0$ then there exists a C^1 vector field $\mathbf{G}$ with $\mathbf{F} = \operatorname{curl} \mathbf{G}$. Hint: Define $\mathbf{G} = (G_1, G_2, G_3)$ by

$$G_1(x, y, z) = \int_0^z F_2(x, y, t)dt - \int_0^y F_3(x, t, 0)dt$$

$$G_2(x, y, z) = -\int_0^z F_1(x, y, t)dt$$

$$G_3(x, y, z) = 0$$