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1 Green's Theorem and Stokes' Theorem

Before proceeding with Green's theorem, we recall the notion of induced positive orientation of a surface. This was discussed at the end of last week's tutorial, and the same discussion is repeated here.

In general, let $S \subseteq \mathbb{R}^3$ be a regular surface. We say $p \in S$ is an interior point of S if there is a local C^1 parametrization $\Phi\colon B_1(0) \subseteq \mathbb{R}^2 \to S$ with $\Phi(0) = p, \Phi_u(0) \times \Phi_v(0) \neq 0$. Any point where this condition fails is called a boundary point, and the collection of such points is denoted ∂S. For our purposes, ∂S will be a piecewise-smooth-regular curve. If S is oriented with normal $\vec{n}$, the positive orientation of ∂S is such that a person walking along ∂S with $\vec{n}$ up, the surface will always be on their left.

There is a special case that is used in the statement of Green's theorem. If a piecewise-smooth surface $S \subseteq \mathbb{R}^3$ is contained in $\mathbb{R}^2$ (i.e., z-coordinate is zero on S), then we talk about its boundary ∂S having positive orientation such that a person walking along ∂S according to the positive direction will always have the inside of S to their left. Here the normal vector is $\hat{k} = (0, 0, 1)$. Unless stated otherwise the orientation of such boundaries is assumed to be positive.

With that out of the way, we can present the elegant Green's theorem which is in some sense a version of the Fundamental Theorem of Calculus in two dimensions.

Theorem 1: Green's Theorem

Let $D \subseteq \mathbb{R}^2$ be a closed region such that ∂D is piecewise C^1. If $\mathbf{F} = (P, Q)$ is a C^1 vector field on D then

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{ds} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

You will frequently encounter this theorem in the following form:

$$\int_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

This is a powerful result that allows us to transform some tricky path integrals into double integrals, or vice versa. It is remarkable that in some sense all of the information conveyed in doing a double integral over the entire region D is equivalent to the information carried by a path integral along the boundary.

Exercise 1. Let C be the positively oriented path along the triangle with vertices $(0, 0), (2, 0), (2, 2)$. Compute

$$\int_C x \sin(x^2) dx + (3e^{y^2} - 2x) dy$$

(Answer: -4, apply Green's theorem.)

Exercise 2. Let C be a positively oriented circle of radius 1 around the origin, and suppose $\phi(x), \psi(y)$ are two smooth functions on $\mathbb{R}$. Which of the following integrals are necessarily zero?

a) $\int_C \phi(y)dx + \psi(x)dy$

b) $\int_C \phi(xy)ydx + \phi(xy)xdy$

c) $\int_C \phi(x)\psi(y)dy$

(Answer: Only (b) is necessarily zero. Apply Green's theorem and come up with simple counterexamples for (a) and (c).)

Let's examine some of the nice corollaries.

Theorem 2: Area Formula

Let $D \subseteq \mathbb{R}^2$ be a closed region such that ∂D is piecewise C^1. Then

$$A(D) = \int_{\partial D} xdy = -\int_{\partial D} ydx = \frac{1}{2}\int_{\partial D} xdy - ydx$$

Theorem 3: Difference of Regions

Let $D_1, D_2 \subseteq \mathbb{R}^2$ be closed regions such that $\partial D_1, \partial D_2$ are piecewise C^1, with orientations induced by D_1, D_2 respectively. Set $D = D_1 \setminus D_2$. If $\mathbf{F} = (P, Q)$ is a C^1 vector field on D such that $Q_x = P_y$ on D then

$$\int_{\partial D_1} \mathbf{F} \cdot \mathbf{ds} = \int_{\partial D_2} \mathbf{F} \cdot \mathbf{ds}$$

Theorem 4: Integration by Parts

Let $D \subseteq \mathbb{R}^2$ be a closed region such that ∂D is piecewise C^1. Let $\phi, \psi \in C^2(D)$. Then

$$\iint_D \phi\Delta\psi dA = -\iint_D \nabla\phi \cdot \nabla\psi dA + \int_{\partial D} \phi\nabla\psi \cdot \mathbf{n}ds$$

$$\iint_D \phi\Delta\psi - \psi\Delta\phi dA = \int_{\partial D} (\phi\nabla\psi - \psi\nabla\phi) \cdot \mathbf{n}ds$$

You may also encounter Green's theorem in a so-called vector form, which can be obtained by identifying $\mathbb{R}^2$ within $\mathbb{R}^3$ as a space with z-coordinate zero. More precisely, assuming all the hypotheses of Green's theorem, treat $\mathbf{F} = (P, Q)$ as $\mathbf{F} = (P, Q, 0)$ and we find

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{ds} = \iint_D (\text{curl}\mathbf{F}) \cdot \mathbf{k}dA$$

In the same vein, we can let $\mathbf{n}$ be the outer normal of ∂D and obtain a divergence form of Green's theorem:

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{n} ds = \iint_D \text{div} \mathbf{F} dA$$

Exercise 3. Verify the divergence theorem for $\mathbf{F}(x, y) = (x, y)$ on the closed unit disk D. Then evaluate the integral of the normal component of $\mathbf{G}(x, y) = (2xy, -y^2)$ around the ellipse $x^2/a^2 + y^2/b^2 = 1$. (Answer: Verify that both sides of the divergence form of Green's theorem are equal in this case, noticing that the unit normal vector $\mathbf{n}$ to ∂D is given by $\mathbf{n}(x, y) = (x, y)$. For the second part, use the divergence form of Green's theorem. Answer is zero.)

Our next spectacular result is Stokes' theorem. It generalizes Green's theorem to three dimensions, and we can even derive Green's theorem from it (although the proof you might see uses Green's theorem).

Theorem 5: Stokes' Theorem

Let $S \subseteq \mathbb{R}^3$ be a compact regular oriented surface with positively oriented piecewise C^2 boundary ∂S. Let $\mathbf{F}$ be a C^1 vector field on S. Then

$$\iint_S \text{curl} \mathbf{F} \cdot \mathbf{dS} = \int_{\partial S} \mathbf{F} \cdot \mathbf{ds}$$

Exercise 4. Evaluate $\iint_S \text{curl} \mathbf{F} \cdot \mathbf{dS}$ where $\mathbf{F} = (z^2, -3xy, x^3y^3)$ and S is the surface defined by the portion of the graph $z = 5 - x^2 - y^2, z \geq 1$, with positive orientation. (Answer: 0, draw a picture of the surface and notice that the boundary is $\{(x, y, 1) \mid x^2 + y^2 = 2\}$, then apply Stokes' theorem.)

Exercise 5. Evaluate the surface integral $\iint_S \text{curl} \mathbf{F} \cdot \mathbf{dS}$ where S is the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$ and $\mathbf{F}(x, y, z) = (x^3, -y^3, 0)$. (Answer: 0, draw a picture of the surface and notice that the boundary is $\{(x, y, 0) \mid x^2 + y^2 = 1\}$, then apply Stokes' theorem.)

For many analogous reasons to Green's theorem, this result is remarkable and has a number of interesting corollaries.

Theorem 6: Common Boundary

Let $S_1, S_2 \subseteq \mathbb{R}^3$ be two oriented surfaces satisfying the conditions of Stokes' theorem such that $\partial S_1 = \partial S_2 = \gamma$. Suppose also that S_1, S_2 induce opposite orientations on γ. Then for any C^1 $\mathbf{F}$:

$$\iint_{S_1} \text{curl} \mathbf{F} \cdot \mathbf{dS} = \iint_{S_2} \text{curl} \mathbf{F} \cdot \mathbf{dS}$$

Theorem 7: No Boundary

If S is a compact oriented surface satisfying the conditions of Stokes' theorem, and S is without boundary, then

$$\iint_S \text{curl}\mathbf{F} \cdot \mathbf{dS} = 0$$

2 Gauss's Divergence Theorem

The last big theorem of vector calculus is the divergence theorem. It gives us a relationship between triple integrals and surface integrals. Suppose $W \subseteq \mathbb{R}^3$ is a bounded solid region with piecewise-smooth boundary $S = \partial W$. We assign an orientation on S so that its normal vector points outward.

Theorem 8: Gauss' Divergence Theorem

Suppose $W \subseteq \mathbb{R}^3$ is a bounded solid region with piecewise-smooth boundary $S = \partial W$. Let $\mathbf{F}$ be a C^1 vector field on W. Then

$$\iiint_W \text{div}\mathbf{F} dV = \iint_S \mathbf{F} \cdot \mathbf{dS}$$

Exercise 6. Let S be the surface of the cube whose main diagonal has endpoints $(0, 0, 0)$ and $(1, 1, 1)$. Let $\mathbf{F}(x, y, z) = (z, y, x)$. What is the value of the flux of $\mathbf{F}$ through S, given that S has outward orientation? (Recall that flux is another term for surface integral over closed surface.) (Answer: 1, draw a picture and apply the divergence theorem, making use of fact that $\iiint_W dV = \text{Vol}(W)$.)

We study some of the corollaries.

Theorem 9: Difference of Surfaces

Suppose $W \subseteq \mathbb{R}^3$ is a bounded solid region satisfying the assumptions of Gauss's theorem, and that $\partial W = S_1 - S_2$. If $\text{div}F = 0$ on W then

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{dS} = \iint_{S_2} \mathbf{F} \cdot \mathbf{dS}$$

Oftentimes our field $\mathbf{F}$ will have undesirable behaviour at certain isolated points, for example division by zero causing blowup. Such points are called singularities, and even if we cannot apply Gauss's theorem directly we can often make use of some results on singularities to manage the problem.

Theorem 10: Singularities

Suppose $W \subseteq \mathbb{R}^3$ is a bounded solid region satisfying the assumptions of Gauss's theorem. Let $p_1, \ldots, p_N \subseteq \operatorname{int} W$. Suppose $\mathbf{F}$ is a C^1 vector field on $W \setminus \{p_1, \ldots, p_N\}$. Let $\delta > 0$ sufficiently small so that $B_\delta(p_i) \subseteq \operatorname{int} W$ for all $i = 1, \ldots, N$. Then

$$\iint_{\partial W} \mathbf{F} \cdot \mathbf{dS} = \sum_{i=1}^{N} \iint_{\partial B_\delta(p_i)} \mathbf{F} \cdot \mathbf{dS}$$

The usual idea is to determine your singularities, isolate them in a number of small balls of radius $\delta > 0$ for some arbitrary δ sufficiently small, and apply Gauss's theorem away from the singularity. Then try to control the behaviour of the leftover integrals as $\delta \to 0$ (remember we chose δ arbitrarily so we are free to vary it!).

Exercise 7. Let $\mathbf{F}$ be a C^1 vector field on $\mathbb{R}^3$. Prove that

$$\operatorname{div}\mathbf{F}(0) = \lim_{r \to 0} \frac{3 \int_{\partial B_r(0)} \mathbf{F} \cdot \mathbf{dS}}{4\pi r^3}$$

(Solution: Notice that $3/(4\pi r^3) = 1/\operatorname{Vol}(B_r(0))$. Then consider the difference

$$\left| \frac{1}{\operatorname{Vol}(B_r(0))} \int_{\partial B_r(0)} \mathbf{F} \cdot \mathbf{dS} - \operatorname{div} F(0) \right|$$

Apply the divergence theorem and use the continuity of $\operatorname{div} F$ to show that the difference goes to zero as $r \to 0$.)