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1 Line Integrals

The definition of the line integral is motivated by calculating the work done by a force field F acting on a particle that moves through a path γ.

Definition 1: Line Integral

Let $\gamma\colon [a,b] \to \mathbb{R}^3$ be a piecewise-smooth path and F a vector field defined in a neighbourhood of the curve $\gamma([a,b]) \subseteq \mathbb{R}^3$. The **line integral** of F along γ is

$$\int_\gamma F \cdot ds := \int_a^b F(\gamma(t)) \cdot \gamma'(t)dt$$

Exercise 1. Compute the path integral of $f(x,y,z) = xyz$ on $\gamma(t) = (\cos t, \sin t, t)$ for $0 \le t \le \pi/2$.

Exercise 2. Evaluate $\int_\gamma ydx + (3y^3 - x)dy + zdz$ for all curves in the family $\gamma(t) = (t, t^n, 0)$, $n = 1, 2, \ldots$, $0 \le t \le 1$.

Exercise 3. Let $\gamma\colon [a,b] \to \mathbb{R}^3$ be a path, and T its unit tangent vector. What is $\int_\gamma T \cdot ds$?

Curves can be reparametrized according to the following definition. A reparametrization changes how the curve is traversed as time evolves (e.g. twice as fast), but the resultant image in $\mathbb{R}^3$ is preserved.

Definition 2: Reparametrization

Suppose $\gamma\colon [a,b] \to \mathbb{R}^3$ is a piecewise-smooth path. A path $\tilde{\gamma}$ is called a **reparametrization** of γ if there is a bijective C^1 map $h\colon [a',b'] \to [a,b]$ such that

$$\tilde{\gamma}(t) = \gamma(h(t)) \ \ \forall t \in [a',b']$$

A reparametrization is called **orientation-preserving** if

$$\tilde{\gamma}(a') = \gamma(a), \ \ \tilde{\gamma}(b') = \gamma(b)$$

A reparametrization is called **orientation-reversing** if

$$\tilde{\gamma}(a') = \gamma(b), \ \ \tilde{\gamma}(b') = \gamma(a)$$

Theorem 1: Orientation of Line Integrals

Suppose $\gamma\colon [a,b] \to \mathbb{R}^3$ is a piecewise-smooth path, and F is a continuous vector field on γ. Let $\tilde{\gamma}$ be a reparametrization of γ. If $\tilde{\gamma}$ is orientiation-preserving then

$$\int_{\tilde{\gamma}} F \cdot ds = \int_{\gamma} F \cdot ds$$

If $\tilde{\gamma}$ is orientiation-reversing then

$$\int_{\tilde{\gamma}} F \cdot ds = -\int_{\gamma} F \cdot ds$$

Note that path integrals of scalar functions are independent of any reparametrization. Now we get a version of the fundamental theorem of calculus for line integrals.

Theorem 2: FTC for Line Integrals

Suppose $F = \nabla f$ for some C^1 function f defined in a neighbourhood of the piecewise-smooth path γ. Then

$$\int_{\gamma} F \cdot ds = f(\gamma(b)) - f(\gamma(a))$$

The condition that F is a gradient of some smooth function is useful, so we give it a name.

Definition 3: Conservative Vector Field

Suppose F is a vector field on $\Omega \subseteq \mathbb{R}^3$. We say F is a **conservative** vector field in Ω if there exists a C^1 function $f\colon \Omega \to \mathbb{R}$ such that $F(x) = \nabla f(x)$ for all $x \in \Omega$.

You might recall from physics courses that work done by conservative forces is independent of the path taken by an object under the influence of the force. This definition and the previous theorem are the proper mathematical formulation of those ideas.

Definition 4: Simple Curves

A curve C is called a **simple curve** if C is the image of a piecewise-smooth path $\gamma\colon [a,b] \to \mathbb{R}^3$ that is bijective on $[a,b]$. The path γ is called a parametrization of C. A **simple closed curve** is the image of a piecewise-smooth path $\gamma\colon [a,b] \to \mathbb{R}^3$ that is bijective on $[a,b)$ and $\gamma(a) = \gamma(b)$.

Given a simple curve $C = \gamma([a,b])$, we can always orient it in one of two ways. If $\gamma'(t) \neq 0$ on $[a,b]$, i.e., γ is regular, then orientation is a continuous choice of the direction of the tangent vector. Going forward, all simple curves are assumed to be oriented. We can now define path integrals along oriented curves by defining the oriented curve integral in terms of the associated path:

$$\int_C F \cdot ds = \int_\gamma F \cdot ds$$

If we denote by C^- the curve C with opposite orientation, then our prior work shows that

$$\int_{C^-} F \cdot ds = -\int_C F \cdot ds$$

We can talk about the algebraic sum of a finite number of oriented curves $C_1, \dots, C_k$.

$$C = C_1 + \cdots + C_k$$

There is no restriction that the endpoints of the C_i agree. Then we can show that

$$\int_C F \cdot ds = \sum_{i=1}^{k} \int_{C_i} F \cdot ds$$

Exercise 4. Let C be the boundary of the square whose vertices are $(0,0), (2,0), (2,2), (0,2)$, oriented counterclockwise. Evaluate each of the following line integrals around C:

a) $\int_C (x+2y)dx + (x-3y^2)dy$

b) $\int_C (x+y)dx + (x-3y^2)dy$

2 Surface Integrals for Vector Fields

Just as we generalized path integrals for scalar functions to line integrals for vector fields, we can do a similar generalization of surface integrals for scalar functions to surface integrals for vector fields. As in the curve case, we need a notion of orientation.

Definition 5: Oriented Surface

A parametrized C^1 regular surface S is called **orientable** if a unit normal vector $\vec{n}$ is assigned at each point on S and $\vec{n}$ is continuous on S.

We remark that if a surface is orientable then it has two orientations: take the unit normal $-\vec{n}$ to obtain the surface S^- with opposite orientation. Some common examples of orientable surfaces and their normal vectors are discussed below.

If $\Phi \colon D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is a parametrization of an oriented regular surface S then the convention is to choose

$$\vec{n}(\Phi(u,v)) = \frac{\Phi_u \times \Phi_v}{\|\Phi_u \times \Phi_v\|}$$

If $S = \{(x, y, z) \mid G(x, y, z) = 0\}$ is a level surface for $G \in C^1, \nabla G \neq 0$ then S is orientable by picking

$$\vec{n}(x, y, z) = \frac{\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|}$$

If $S = \left\{(x, y, g(x, y)) \mid (x, y) \in D \subseteq \mathbb{R}^2\right\}$ is a graph then S is orientable with normal vector

$$\vec{n} = \frac{(-g_x, -g_y, 1)}{\sqrt{1 + g_x^2 + g_y^2}}$$

Definition 6: Surface Integral

Let S be an oriented surface in $\mathbb{R}^3$, and F a vector field defined on S. The **surface integral** of F over S is

$$\int_S F \cdot dS := \iint_S F \cdot \vec{n} dS$$

We have analogous reparametrization definitions and results.

Definition 7: Reparametrization

Suppose $\Phi\colon D \subseteq \mathbb{R}^2 \to S$ is a parametrization of S. A map $\tilde{\Phi}\colon \tilde{D} \subseteq \mathbb{R}^2 \to S$ is called a **reparametrization** of S if there is a bijective C^1 map $h\colon \tilde{D} \to D$ such that $h^{-1} \in C^1$ and

$$\tilde{\Phi}(s, t) = \Phi(h(s, t)) \;\; \forall (s, t) \in \tilde{D}$$

A reparametrization is called **orientation-preserving** if

$$\frac{\tilde{\Phi}_s \times \tilde{\Phi}_t}{\|\tilde{\Phi}_s \times \tilde{\Phi}_t\|}(s, t) = \frac{\Phi_u \times \Phi_v}{\|\Phi_u \times \Phi_v\|}(h(s, t))$$

A reparametrization is called **orientation-reversing** if

$$\frac{\tilde{\Phi}_s \times \tilde{\Phi}_t}{\|\tilde{\Phi}_s \times \tilde{\Phi}_t\|}(s, t) = -\frac{\Phi_u \times \Phi_v}{\|\Phi_u \times \Phi_v\|}(h(s, t))$$

If the reparametrization is orientation-preserving we have

$$\iint_{\Phi(D)} F \cdot dS = \iint_{\tilde{\Phi}(\tilde{D})} F \cdot dS$$

If the reparametrization is orientation-reversing we have

$$\iint_{\Phi(D)} F \cdot dS = -\iint_{\tilde{\Phi}(\tilde{D})} F \cdot dS$$

In particular, if S is an oriented surface we have

$$\int_{S^-} F \cdot dS = -\int_S F \cdot dS$$

We conclude with a discussion about orientation of boundary surfaces. If piecewise-smooth $S \subseteq \mathbb{R}^3$ is contained in $\mathbb{R}^2$ (i.e., z-coordinate is zero on S), then we talk about its boundary ∂S having positive orientation such that a person walking along ∂S according to the positive direction will always have the inside of S to their left. Here the normal vector is $\hat{k} = (0, 0, 1)$. Unless stated otherwise the orientation of such boundaries is assumed to be positive.

In general, let $S \subseteq \mathbb{R}^3$ be a regular surface. We say $p \in S$ is an interior point of S if there is a local C^1 parametrization $\Phi\colon B_1(0) \subseteq \mathbb{R}^2 \to S$ with $\Phi(0) = p, \Phi_u(0) \times \Phi_v(0) \neq 0$. Any point where this condition fails is called a boundary point, and the collection of such points is denoted ∂S. For our purposes, ∂S will be a piecewise-smooth-regular curve. If S is oriented with normal $\vec{n}$, the positive orientation of ∂S is such that a person walking along ∂S with $\vec{n}$ up, the surface will always be on their left.

Exercise 5. Calculate $\int_S F \cdot dS$ where $F(x, y, z) = (x, y, z^4)$ where S is the surface of the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$ including the disk $x^2 + y^2 \leq 9$ with $z = 0$. Assume S to be positively oriented.