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1 Triple Integrals

The development and definitions for triple integrals are completely analogous to the double integral. The key idea is that partitions are now collections of three ordered sets, the rectangles R_{ijk} defined by these partitions are three-dimensional boxes, the function f is defined on a rectangular region $W \subseteq \mathbb{R}^3$, and the notation now has three integrals rather than two.

Many of the theorems from last Friday's tutorial remain true with slight modifications to notation. The refinement bounds in Theorem 1 remain true, as do the integral properties in Theorem 3. In particular, triple integrals are still linear in the sense that

$$\iiint_W (af + bg)dV = a \iiint_W f dV + b \iiint_W g dV$$

and they are monotone in the sense that if $f \geq g$ on W then

$$\iiint_W f dV \geq \iiint_W g dV$$

We take a minute to record a useful lemma on checking if a function is Riemann-integrable. It is also valid for the double integrals we discussed last time (just replace W by R).

Theorem 1: Integrability Criterion

Let $f : W \to \mathbb{R}$ be bounded. Then f is Riemann-integrable over W if and only if for all $\varepsilon > 0$ there exists a partition $\mathcal{P}$ such that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$$

We also mention the analogue of Theorem 2:

Theorem 2: Riemann-integrable Functions

Let $f : W \to \mathbb{R}$ be bounded. If f is continuous then f is Riemann-integrable over W.
More generally, if $\Gamma_i \subseteq W$, $i = 1, \ldots, k$ are graphs of continuous real-valued functions on bounded regions in $\mathbb{R}^2$ (i.e., surfaces) and f is continuous on $W \setminus \bigcup_{i=1}^k \Gamma_i$ then f is Riemann-integrable over W.

The extension of the integral to non-rectangular regions follows the same idea as the double integral. For a general bounded region W, find a rectangular region $\tilde{W} \supseteq W$ and define an auxiliary function $\tilde{f}$ that agrees with f on W and is 0 otherwise. Then the integral of f on W is defined to be the integral of $\tilde{f}$ on $\tilde{W}$ (if it exists).

Simple regions in $\mathbb{R}^3$ are also defined in a natural way that reminds us of the two-dimensional case.

Definition 1: Simple Regions

Let $W \subseteq \mathbb{R}^3$. Then W is called a **z-simple region** if there is an elementary region $D \subseteq \mathbb{R}^2$ and continuous functions $\eta_1, \eta_2 \colon D \to \mathbb{R}$ such that

$$W = \{(x, y, z) \mid \eta_1(x, y) \leq z \leq \eta_2(x, y) \; \forall (x, y) \in D\}$$

Similarly, **x-simple** and **y-simple** regions are defined by changing the coordinates in the above definition appropriately. D is called a **simple region** if it is x-simple, y-simple, and z-simple. Any of these regions is called an **elementary region**.

Now we can introduce Fubini's theorem for triple integrals. Unsurprisingly, it looks like Fubini's theorem for double integrals.

Theorem 3: Fubini's Theorem

Let $f \colon W \to \mathbb{R}$ with W bounded and piecewise-smooth (its boundary is a union of finitely many graphs of smooth functions). Suppose the conditions of Theorem 2 hold. Then f is integrable on W and if furthermore W is z-simple and $\int_{\eta_1(x,y)}^{\eta_2(x,y)} f(x, y, z)dz$ exists for all $(x, y) \in D$ then

$$\iiint_W f dV = \iint_D \left(\int_{\eta_1(x,y)}^{\eta_2(x,y)} f(x, y, z)dz \right) dA$$

Analogous formulas hold if W is x-simple or y-simple.

Exercise 1. Evaluate $\iiint_W x^2 \cos z dxdydz$ where W is the region bounded by the planes $z = 0, z = \pi, y = 0, y = 1, x = 0, x + y = 1$.

Exercise 2. Evaluate $\iiint_W (1 - z^2)dxdydz$ where W is the pyramid with top vertex at $(0, 0, 1)$, and base vertices at $(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0)$.

Exercise 3. Find the appropriate limits $\phi_1(x), \phi_2(x), \eta_1(x, y), \eta_2(x, y)$to write

$$\iiint_W f dV = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\eta_1(x,y)}^{\eta_2(x,y)} f(x, y, z)dzdydx$$

where $W = \left\{(x, y, z) \mid x^2 + y^2 \leq 1, z \geq 0, x^2 + y^2 + z^2 \leq 4\right\}$.

Let us introduce some notation briefly: Suppose $x = x(u,v,w), y = y(u,v,w), z = z(u,v,w)$ are C^1 functions of u, v, w. Then we write

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} := \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

Theorem 4: Change of Variables Theorem

Let W, W^* be bounded piecewise-smooth regions in $\mathbb{R}^3$, and let $T\colon W^* \to W$ be a bijective C^1 map given by $x = x(u,v,w), y = y(u,v,w), z = z(u,v,w)$. Then for any integrable function $f\colon W \to \mathbb{R}$:

$$\iiint_W f(x,y,z)dxdydz = \iiint_{W^*} f(u,v,w) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw$$

Exercise 4. Evaluate $\iiint_W \frac{dxdydz}{(x^2+y^2+z^2)^{3/2}}$ where W is the region bounded by the two spheres $x^2 + y^2 + z^2 = a^2, x^2 + y^2 + z^2 = b^2$ with $0 < b < a$.

2 Surface Integrals

A surface S is the graph of a C^1 function $g\colon D \subseteq \mathbb{R}^2 \to \mathbb{R}$ with D piecewise smooth and bounded:

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid z = g(x,y), (x,y) \in D\}$$

Geometric arguments from class lead us to natural definition of area for such surfaces

$$A(S) = \iint_D \sqrt{1 + g_x^2 + g_y^2} dxdy$$

We can be much more general than this, and we encapsulate this generality in the following definition.

Definition 2: Parametrized Surface

A **parametrization of a surface** is a map $\Phi\colon D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, and the **parametrized surface** S is the image $\Phi(D)$:

$$S = \{(x(u,v), y(u,v), z(u,v) = \Phi(u,v) \mid (u,v) \in D\}$$

We say S is a C^1 surface if Φ is C^1, and S is a **regular** surface if $\Phi_u \times \Phi_v \neq 0$ for all $(u,v) \in D$.

Again, geometric considerations lead us to define the tangent vectors $T_u = \Phi_u$, $T_v = \Phi_v$, a normal vector to the surface $T_u \times T_v$, and the area element of the surface S as $dS = \|T_u \times T_v\|$, from which we define

$$A(S) = \iint_D \|T_u \times T_v\| dA$$

It is a good idea to verify that we can recover the special case when S is a graph using this formula.

Exercise 5. Find an expression for a unit vector normal to the surface

$$x = \cos v \sin u, y = \sin v \sin u, z = \cos u$$

at the image of a point $(u, v) \in [0, \pi] \times [0, 2\pi]$. What is this surface?

Exercise 6. Find the area of the surface defined by $x + y + z = 1, x^2 + 2y^2 \leq 1$.

Now we can introduce another key construct of vector calculus - the surface integral.

Definition 3: Surface Integral

Let S be a parametrized surface and $f\colon S \to \mathbb{R}$ a bounded function. We define the **surface integral of f on S** to be

$$\iint_S f dS := \iint_D f(\Phi(u, v))\|T_u \times T_v\| dudv$$

Exercise 7. Let S be the surface defined by $\Phi(u, v) = (u + v, u - v, uv)$. Show that the image of Φ is in the graph of the surface $4z = x^2 - y^2$. Then evaluate $\iint_S xdS$ for all points on the graph S over $x^2 + y^2 \leq 1$.