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1 The Riemann Integral

In one dimension, the integral of a non-negative real-valued function can be interpreted as the area between the graph and the x-axis. In two dimensions, the integral of a non-negative real-valued function can be interpreted as the volume between the graph and the xy-plane. We need some technical definitions to make sense of these notions rigorously.

Definition 1: Partition

Let $R = [a, b] \times [c, d]$ be a rectangle in $\mathbb{R}^2$. A **partition** of R is two ordered sets

$$\mathcal{P} = \{x_0, x_1, \dots, x_n; y_0, y_1, \dots, y_m\}$$

such that

$$a = x_0 < x_1 < \cdots < x_n = b, c = y_0 < y_1 < \cdots < y_m = d$$

If we have two partitions,

$$\mathcal{P} = \{x_0, x_1, \dots, x_n; y_0, y_1, \dots, y_m\}$$

$$\tilde{\mathcal{P}} = \{\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_k; \tilde{y}_0, \tilde{y}_1, \dots, \tilde{y}_l\}$$

we say that $\tilde{\mathcal{P}}$ is a **refinement** of $\mathcal{P}$ if

$$\{x_0, x_1, \dots, x_n\} \subseteq \{\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_k\}$$

$$\{y_0, y_1, \dots, y_m\} \subseteq \{\tilde{y}_0, \tilde{y}_1, \dots, \tilde{y}_l\}$$

We also denote by $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$. These are the sub-rectangles that make up the partition. Throughout this section R will denote the rectangle $[a, b] \times [c, d]$.

Definition 2: Lower and Upper Estimates

Let $f \colon R \to \mathbb{R}$ be bounded. Then for any partition $\mathcal{P}$ of R, define

$$L(f, \mathcal{P}) = \sum_{j=1}^{m} \sum_{i=1}^{n} \inf_{x \in R_{ij}} f(x) \Delta x_i \Delta y_j$$

$$U(f, \mathcal{P}) = \sum_{j=1}^{m} \sum_{i=1}^{n} \sup_{x \in R_{ij}} f(x) \Delta x_i \Delta y_j$$

The next result essentially says that as you take your partitions finer and finer, your lower and upper estimates of the integral will get better and approach each other. However, they may not approach each other entirely, and such functions are not Riemann-integrable by definition.

Theorem 1: Refinement Bounds

Let $f\colon R \to \mathbb{R}$ be bounded, and let $\mathcal{P}, \tilde{\mathcal{P}}$ be partitions of R such that $\tilde{\mathcal{P}}$ is a refinement of $\mathcal{P}$. Then

$$L(f,\mathcal{P}) \leq L(f,\tilde{\mathcal{P}}) \leq U(f,\tilde{\mathcal{P}}) \leq U(f,\mathcal{P})$$

Finally we can define the Riemann integral for functions whose lower and upper estimates do converge towards each other with equality in the limit.

Definition 3: Riemann Integral

Let $f\colon R \to \mathbb{R}$ be bounded. Define

$$\overline{\iint_R} f(x)dA = \inf\{U(f,\mathcal{P}) \mid \mathcal{P} \text{ partition of } R\}$$

$$\underline{\iint_R} f(x)dA = \sup\{L(f,\mathcal{P}) \mid \mathcal{P} \text{ partition of } R\}$$

We say f is **Riemann-integrable** over R if $\overline{\iint_R} f(x)dA = \underline{\iint_R} f(x)dA$ in which case we just write

$$\iint_R f(x)dA = \overline{\iint_R} f(x)dA = \underline{\iint_R} f(x)dA$$

What functions are Riemann-integrable? It turns out that continuous functions and functions that are continuous everywhere except for some well-behaved subsets are Riemann-integrable.

Theorem 2: Riemann-Integrable Functions

Let $f\colon R \to \mathbb{R}$ be bounded. If f is continuous, then f is Riemann-integrable. More generally, if $\gamma_i \subseteq R, i = 1, \dots, k$ are the curves of continuous functions (either over the x-axis or y-axis) such that f is continuous on $R \setminus \bigcup_{i=1}^k \gamma_i$ then f is Riemann-integrable.

The integral satisfies the following familiar properties.

Theorem 3: Integral Properties

Suppose f, g are Riemann-integrable over R. Then

a) $f + g$ is Riemann-integrable over R and

$$\iint_R (f+g)dA = \iint_R f dA + \iint_R g dA$$

b) For all $c \in \mathbb{R}$, cf is Riemann-integrable over R and

$$\iint_R cf dA = c \iint_R f dA$$

c) If $f(x,y) \geq g(x,y)$ on R then

$$\iint_R f dA \geq \iint_R g dA$$

d) If $R = R_1 \cup \cdots \cup R_k$ with $R_i, i = 1, \ldots, k$ pairwise disjoint rectangles then

$$\iint_R f dA = \sum_{i=1}^{k} \iint_{R_i} f dA$$

Exercise 1. If $f(x,y) = e^{\sin(x+y)}$ on $D = [-\pi, \pi] \times [-\pi, \pi]$, show that

$$\frac{1}{e} \leq \frac{1}{4\pi^2} \iint_D f(x,y) dA \leq e$$

Now we wish to extend the integral to functions defined on more general regions than rectangles.

Definition 4: Integral on Bounded Regions

Let $D \subseteq \mathbb{R}^2$ be a bounded region and $f \colon D \to \mathbb{R}$ be bounded. Let $R \supseteq D$ be a rectangle. Define a function on R:

$$f_R^*(x,y) = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & (x,y) \notin D \end{cases}$$

We say f is Riemann-integrable over D if there is a rectangle $R \supseteq D$ such that f_R^* is Riemann-integrable on R and we define

$$\iint_D f dA = \iint_R f_R^* dA$$

Exercise 2. Let f be continuous, $f \geq 0$ on R. Show that if $\iint_R f dA = 0$ then $f = 0$ on R.

At this point we define some nice regions in the plane that we can often compute integrals over without too much trouble.

Definition 5: Simple Regions

Let $D \subseteq \mathbb{R}^2$. Then D is called a **y-simple region** if there exist continuous functions $\phi_1, \phi_2 \colon [a, b] \to \mathbb{R}$ such that

$$D = \{(x, y) \mid a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)\}$$

Similarly, D is called an **x-simple region** if there exist continuous functions $\psi_1, \psi_2 \colon [c, d] \to \mathbb{R}$ such that

$$D = \{(x, y) \mid c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y)\}$$

D is called a **simple region** if it is both x-simple and y-simple. Any of these regions is called an **elementary region**.

Theorem 4: Integral on Elementary Regions

Let $D = \bigcup_{i=1}^k D_i$ with each D_i, $i = 1, \ldots, k$ an elementary region. Suppose $\operatorname{int} D_i \cap \operatorname{int} D_j = \emptyset$ for all $i \neq j$. Suppose $\gamma_i \subseteq D$, $i = 1, \ldots, m$ are curves of continuouse functions (either over the x-axis or y-axis) and suppose that f is continuous on $D \setminus \bigcup_{i=1}^m \gamma_i$. Then f is integrable on D.

Double integrals are nice in theory but computing them from the definition is difficult to do by hand. We can use Fubini's theorem to write many double integrals as iterated integrals, and then use our knowledge of one-dimensional calculus (e.g. FTC) to carry out the integration.

Theorem 5: Fubini's Theorem

Let $f \colon R \to \mathbb{R}$ be Riemann-integrable over R and suppose for each $x \in [a, b]$ that $\int_c^d f(x, y)dy$ exists. Then $\int_a^b \int_c^d f(x, y)dydx$ exists and

$$\int_a^b \int_c^d f(x, y)dydx = \iint_R f dA$$

Exercise 3. Compute the volume of the solid bounded by the surface $z = \sin y$, the planes $x = 1, x = 0, y = 0, y = \pi/2$, and the xy-plane.

Of course, the symmetric statement interchanging x and y is also true. We can cover all of these cases in the following general theorem:

Theorem 6: Iterated Integrals on Elementary Regions

Let $f\colon D \to \mathbb{R}$ where f, D satisfy the assumptions of Theorem 4. Then

a) If D is y-simple and for every $x \in [a,b]$, $\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy$ exists, then $\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dydx$ exists and

$$\iint_D f dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dydx$$

b) If D is x-simple and for every $y \in [c,d]$, $\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx$ exists, then $\int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dxdy$ exists and

$$\iint_D f dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dxdy$$

Exercise 4. Show that $\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2} dydx = \frac{\pi}{4}$ but that the iterated integral in the opposite order is equal to $-\frac{\pi}{4}$. Does this contradict Fubini's theorem?

Exercise 5. Evaluate $\int_0^1 \int_{\sqrt{y}}^1 e^{x^3} dxdy$.

Exercise 6. Evaluate $\iint_D x dA$ where D is the interior of the triangle with vertices $(-3,0), (6,0), (0,3)$.

We also have a mean value theorem for double integrals.

Theorem 7: Mean Value Theorem for Double Integrals

Suppose $f\colon D \to \mathbb{R}$ is continuous on an elementary region $D \subseteq \mathbb{R}^2$. Then there is a point $(x_0, y_0) \in D$ such that

$$\iint_D f dA = f(x_0, y_0)A(D)$$

where $A(D)$ is the area of D defined by

$$A(D) = \iint_D 1 dA$$

Let us introduce some notation briefly: Suppose $x = x(u,v), y = y(u,v)$ are C^1 functions of u, v. Then we write

$$\frac{\partial(x,y)}{\partial(u,v)} := \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

The change of variables theorem allows us to transform the integrand and domain of integration simultaneously to obtain a simpler equivalent integral.

Theorem 8: Change of Variables Theorem

Let D, D^* be elementary regions in $\mathbb{R}^2$, and let $T\colon D^* \to D$ be a bijective C^1 map given by $x = x(u,v), y = y(u,v)$. Then for any integrable function $f\colon D \to \mathbb{R}$:

$$\iint_D f(x,y)dxdy = \iint_{D^*} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

Exercise 7. Calculate $\iint_R (x+y)^2 e^{x-y} dA$ where R is the region bounded by $x-y=-1, x-y=1, x+y=1, x+y=4$.