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1 Curves and Path Integrals

We will begin by studying functions from intervals in $\mathbb{R}$ to a higher dimensional space $\mathbb{R}^n$. Such functions have the geometric interpretation as curves or paths in $\mathbb{R}^n$, take for example the trajectory that a particle takes over a period of time.

Definition 1: C^1 Curve

Let $\gamma\colon [a,b] \to \mathbb{R}^n$. We say that $\gamma \in C^1([a,b])$ if γ is C^1 on (a,b) and

$$\gamma'(a+) := \lim_{t\to a+} \frac{\gamma(t)-\gamma(a)}{t-a}$$

$$\gamma'(b-) := \lim_{t\to b-} \frac{\gamma(t)-\gamma(b)}{t-b}$$

exist and $\lim_{t\to a+}\gamma'(t) = \gamma'(a+)$, $\lim_{t\to b-}\gamma'(t) = \gamma'(b-)$.

An equivalent formulation is that γ is the restriction of a C^1 map from some open interval $(A,B) \supseteq (a,b)$ with $A < a, b < B$.

We can write any such γ as $\gamma(t) = (x_1(t), \dots, x_n(t))$ and taking the derivative gives us the **tangent vector**:

$$\gamma'(t) = (x_1'(t), \dots, x_n'(t))$$

Continuing our particle analogy, the tangent vector represents the velocity vector of the particle and its norm is the speed.

A natural problem is to measure the length of a curve, and using an argument based on polygonal paths it becomes clear that the proper definition is via an integral.

Definition 2: Length of a Curve

Let $\gamma\colon [a,b] \to \mathbb{R}^n$ belong to $C^1([a,b])$. We define the length element $ds = \|\gamma'(t)\|dt$. Then the **length** of the curve γ is

$$L(\gamma) := \int_a^b \|\gamma'(t)\|dt = \int_\gamma ds$$

For a C^1 function $f\colon [a,b] \to \mathbb{R}$ we can realize its graph as a curve γ defined by $\gamma(t) = (t, f(t))$. Then using the formula above we find

$$L(\gamma) = \int_a^b \sqrt{1+(f'(t))^2}dt$$

We can enlarge our class of curves by ignoring issues of differentiability at finitely many points, as long as we still maintain continuity.

Definition 3: Curve Properties

Suppose $\gamma\colon [a,b] \to \mathbb{R}^n$ is continuous. We say γ is:

a) **piecewise** C^1 if there exists $a = a_0 < a_1 < \cdots < a_m = b$ such that $\gamma \in C^1([a_{i-1}, a_i])$ for all $i = 1, \ldots, m$.
b) **regular** if $\gamma \in C^1([a,b])$ and $\gamma'(t) \neq 0$ for all $t \in [a,b]$.
c) **piecewise smooth** if there exists $a = a_0 < a_1 < \cdots < a_m = b$ such that γ is regular in each $[a_{i-1}, a_i]$ for all $i = 1, \ldots, m$.

The next theorem gives us some intuition for why regular curves are nice. It tells us that regular curves look locally like graphs, with all coordinates dependent on only one of the coordinates.

Theorem 1: Regular Curve Property

If $\gamma\colon [a,b] \to \mathbb{R}^n$ is regular then there exists $a = t_0 < t_1 < \cdots < t_N = b$ such that for all $i = 1, \ldots, N$, on each $[t_{i-1}, t_i]$ there is an index j_i and a map $t \mapsto x_{j_i}(t)$ that is C^1 and invertible such that

$$\gamma(t) = (x_1(x_{j_i}(t)), \ldots, x_{j_i-1}(x_{j_i}(t)), x_{j_i}(t), \ldots, x_n(x_{j_i}(t)))$$

That is, γ is a graph over x_{j_i}.

Now we can generalize our definition of length to piecewise C^1 curves:

Definition 4: Length of a Piecewise C^1 Curve

Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be piecewise C^1 on $[a,b]$. Then the **length** of the curve γ is

$$L(\gamma) := \sum_{i=1}^{m} \int_{a_{i-1}}^{a_i} \|\gamma'(t)\| dt$$

where the partition corresponds to the one given by the piecewise C^1 property.

Finally we can introduce a fundamental tool of vector calculus: the path integral (also called line integral).

Definition 5: Path Integral

Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be piecewise C^1 on $[a,b]$. Suppose f is a continuous function defined along the image of γ. Then the **path integral** of f along γ is

$$\int_\gamma f\,ds := \int_a^b f(\gamma(t))\|\gamma'(t)\|dt = \sum_{i=1}^m \int_{a_{i-1}}^{a_i} f(\gamma(t))\|\gamma'(t)\|dt$$

where the partition corresponds to the one given by the piecewise C^1 property.

For the remainder of this section assume that $\gamma\colon [a,b] \to \mathbb{R}^3$ is a C^3 regular curve. Then its arc-length function

$$s(t) := \int_a^t \|\gamma'(u)\|du$$

is strictly increasing so the relationship between t and s is invertible. This allows us to write $t = h(s)$ for some function h. Then we can define $\tilde{\gamma}(s) := \gamma(h(s))$ and check that $\|\tilde{\gamma}'(s)\| = 1$ for all s. The function $\tilde{\gamma}(s)$ is called the arc-length parametrization of γ.

Now we know that we can parametrize by arc-length, let us assume going forward that we have already done so, i.e. γ is a function of s and satisfies $\|\gamma'(s)\| = 1$.

Then the tangent vector $T(s) = \gamma'(s)$ is a unit vector and $\gamma''(s) = T'(s)$ is orthogonal to $T(s)$.

There are a handful of related quantities and functions that we can associate with such curves. We collect them in the following definition.

Definition 6: Geometry of Curves

Suppose $\gamma\colon [a,b] \to \mathbb{R}^3$ is a C^3 regular curve parametrized by arc-length.

a) $\kappa(s) := \|T'(s)\|$ is the **curvature** of γ.
b) If $\kappa(s) > 0$ define the **principal normal vector** $N(s) = \frac{T'(s)}{\|T'(s)\|}$.
c) $B(s) := T(s) \times N(s)$ is the **binormal vector**.

It can be checked that $B'(s) = -\tau(s)N(s)$ and the function τ is called the **torsion** of γ.

2 Vector Fields

Vector fields are just functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. We will focus on differentiable vector fields and various operations we can do with them.

Definition 7: Vector Field

A function $F\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is called a **vector field**. To every point $x \in \Omega$ it assigns another vector $F(x) \in \mathbb{R}^n$.

Take any differentiable function $f\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$. Then its gradient $\nabla f\colon \Omega \to \mathbb{R}^n$ is a vector field called the **gradient vector field**. We have to get through several definitions before we can do anything interesting.

Definition 8: Flow Lines

Let F be a vector field on $\Omega \subseteq \mathbb{R}^n$. A **flow line** for F is a curve γ mapping into Ω such that

$$\gamma'(t) = F(\gamma(t))$$

That is, F yields the velocity field of the path γ.

For any vector field F we can obtain a meaningful scalar function on Ω called the divergence of F.

Definition 9: Divergence

Let $F = (F_1, \ldots, F_n)$ be a C^1 vector field on $\Omega \subseteq \mathbb{R}^n$. The **divergence** of F is the scalar function

$$\operatorname{div} F :=: \nabla \cdot F := \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$$

Naturally we wonder if for any vector field F we can obtain a different meaningful vector field on Ω too. The problem is subtle, so we restrict our attention to $\mathbb{R}^3$ where the answer is affirmative and this field is called the curl of F.

Definition 10: Curl

Let $F = (F_1, F_2, F_3)$ be a C^1 vector field on $\Omega \subseteq \mathbb{R}^3$. The **curl** of F is the vector field

$$\operatorname{curl} F :=: \nabla \times F := \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right)$$

When you need to know the definition of curl, just remember the determinant formula for cross product. The curl is not a literal cross product, but you can think of it as a symbolic cross product that uses differentiation operators.

$$\operatorname{curl} F = \nabla \times F = \det \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{pmatrix} = \det \begin{pmatrix} i & \frac{\partial}{\partial x_1} & F_1 \\ j & \frac{\partial}{\partial x_2} & F_2 \\ k & \frac{\partial}{\partial x_1} & F_3 \end{pmatrix}$$

The physical significance of these quantities and fields will become more clear after we cover the fundamental theorems of calculus in $\mathbb{R}^2$ and $\mathbb{R}^3$.

We say that a vector field F is **incompressible** if $\operatorname{div} F = 0$, and that it is **irrotational** if $\operatorname{curl} F = 0$. There are various identities for the divergence and curl operators that you can prove. Some are in the lecture slides, and it is a good exercise of using the definitions to verify them.

We conclude with two special results for C^2 functions.

Theorem 2: Gradient is Irrotational

Let $f\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be C^2. Then $\operatorname{curl} \nabla f = 0$.

Theorem 3: Curl is Incompressible

Let $F\colon \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 vector field. Then $\operatorname{div} \operatorname{curl} F = 0$.

3 Exercises

1. (Marsden & Tromba 4.1 #11) Determine which of the following paths are regular.
   a) $\gamma(t) = (\cos t, \sin t, t)$
   b) $\gamma(t) = (t^3, t^5, \cos t)$
   c) $\gamma(t) = (t^2, e^t, 3t + 1)$
2. (Marsden & Tromba 4.2 #13) Let γ be the path $\gamma(t) = (2t, t^2, \log t)$ defined for $t > 0$. Find the arc-length of γ between the two points $(2, 1, 0)$ and $(4, 4, \log 2)$.
3. (Marsden & Tromba 4.2 #14) Find the arc-length function for the curve $\gamma(t) = (\cosh t, \sinh t, t)$.
4. (Marsden & Tromba 4.2 #18) Show that any line $l(t) = x_0 + tv$ has zero curvature, where v is a unit vector.
5. (Straight lines are shortest) Let $\gamma\colon (a, b) \to \mathbb{R}^3$ be a C^1 curve. Let $[c, d] \subseteq (a, b)$, and set $\gamma(c) = p$, $\gamma(d) = q$.
   a) Show that for any constant unit vector v

$$(q - p) \cdot v = \int_c^d \gamma'(t) \cdot v dt \leq \int_c^d \|\gamma'(t)\| dt$$

b) Set $v = \frac{q-p}{\|q-p\|}$ and show that

$$\|\gamma(d) - \gamma(c)\| \leq \int_c^d \|\gamma'(t)\| dt$$

that is, the curve of shortest length from $\gamma(c)$ and $\gamma(d)$ is the straight line joining these points.

6. (Marsden & Tromba 4.3 #18) Show that $\gamma(t) = (\sin t, \cos t, e^t)$ is a flow line for $F(x, y, z) = (y, -x, z)$.

7. (Marsden & Tromba 4.3 #21) Let $F(x, y, z) = (yz, xz, xy)$. Find a function $f\colon \mathbb{R}^3 \to \mathbb{R}$ such that $F = \nabla f$. Do the same for $F(x, y, z) = (x, y, z)$.

8. (Marsden & Tromba 4.4 #3) Find the divergence of $F(x, y, z) = (x, y + \cos x, z + e^{xy})$.

9. (Marsden & Tromba 4.4 #14) Find the curl of $F(x, y, z) = (yz, xz, xy)$.

10. (Marsden & Tromba 4.4 #21) Let $F(x, y, z) = (x^2, x^2 y, z + zx)$. Show that $\nabla \cdot (\nabla \times F) = 0$. Does there exist $f\colon \mathbb{R}^3 \to \mathbb{R}$ such that $F = \nabla f$?

11. (Marsden & Tromba 4.4 #28) Prove that $\operatorname{div}(\nabla f \times \nabla g) = 0$.