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1 Optimization

Last time we talked about critical points and their relationship with maxima and minima. In particular recall that the Second Derivative Test in higher dimensions tells us that positive/negative definiteness of the Hessian at a critical point guarantees us a minimum or maximum respectively, but the conclusion need not hold if we have only *semi*-definiteness. There is one more notable phenomenon that can occur at a critical point.

Definition 1: Saddle Point

Let $f\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in \Omega$ a critical point of f. If x_0 is neither a maximum nor a minimum we say that x_0 is a **saddle point**. In particular, for all $r > 0$ we can find points x_1 and x_2 in $B_r(x_0)$ with

$$f(x_1) < f(x_0) < f(x_2)$$

It is useful to know when a critical point is a saddle point. The following theorem gives us a sufficient condition.

Theorem 1: Saddle Point Existence

Let $f\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be C^3 and x_0 a critical point of f. If $\det(H_f(x_0)) \neq 0$ and $H_f(x_0)$ is neither positive definite nor negative definite then x_0 is a saddle point.

As we mentioned last time, checking if a general symmetric matrix is positive or negative definite can be challenging (for example, naively you could compute all of its eigenvalues). So to make our lives easier let's remember a trick for the case $n = 2$: If A is a 2×2 matrix and $\det(A) < 0$ then A is neither positive definite nor negative definite. We summarize the special case $n = 2$ in the following theorem:

Theorem 2: Second Derivative Test in $\mathbb{R}^2$

Let $f\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be C^2 and x_0 a critical point of f. Suppose $H_f(x_0) = \begin{pmatrix} f_{xx}(x_0) & f_{xy}(x_0) \\ f_{yx}(x_0) & f_{yy}(x_0) \end{pmatrix}$. Then

1. If $\det(H_f(x_0)) < 0$ then x_0 is a saddle point.
2. If $\det(H_f(x_0)) > 0$ and $f_{xx}(x_0) > 0$ then x_0 is a local minimum.
3. If $\det(H_f(x_0)) > 0$ and $f_{xx}(x_0) < 0$ then x_0 is a local maximum.
4. If $\det(H_f(x_0)) = 0$ the test is inconclusive.

We know from calculus that a continuous function on a closed and bounded set attains its extreme values. Our study of critical points allows us to deal with extreme points that lie "inside" of Ω (in its interior), but we need to be careful when searching for extreme points that lie on the boundary.

There are two main strategies for dealing with the function on the boundary.

1. If the boundary is sufficiently simple, for example a rectangle or triangle in $\mathbb{R}^2$ that is piecewise linear, then we can restrict our function to the lines defining the boundary and reduce it to a problem of 1-dimensional optimization.
2. If the boundary is given by the form $\{x \in \mathbb{R}^n \mid g(x) = 0\}$ for sufficiently nice g (i.e. $g \in C^1$) then we use the method of **Lagrange multipliers**

We will see examples of both approaches in the exercises. For now we study the theory behind Lagrange multipliers.

Theorem 3: Lagrange Multiplier Theorem

Let f and g be C^1 functions from $\mathbb{R}^n$ to $\mathbb{R}$. Suppose x_0 belongs to the level set

$$\Gamma = \{x \in \mathbb{R}^n \mid g(x) = 0\}$$

In other words, $g(x_0) = 0$. Suppose also that $Dg(x_0) \neq 0$. If the restricted function $\tilde{f} := f|_\Gamma$ has a maximum or minimum value at x_0 then there exists a constant $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_0) = \lambda \nabla g(x_0)$$

While the statement of this theorem is sophisticated, the key idea is that if we are solving an optimization problem of the form

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & g(x) = 0 \end{aligned} \tag{1}$$

then we are searching for optimizers of the function $\tilde{f}$ and the theorem tells us that we can try and study the system of equations

$$\nabla f(x) = \lambda \nabla g(x)$$

$$g(x) = 0$$

to solve our problem. There is a more general version of this theorem for multiple constraints. We state it here for completeness.

Theorem 4: General Lagrange Multiplier Theorem

Let f and g_i, $i = 1, \ldots, k$ be C^1 functions from $\Omega \subseteq \mathbb{R}^n$ to $\mathbb{R}$. Let $x_0 \in \Omega$ and define

$$\Gamma = \{x \in \Omega \mid g_i(x) = g_i(x_0)\ \forall i = 1, \ldots, k\}$$

Suppose also that $\nabla g_1(x_0), \ldots, \nabla g_k(x_0)$ are linearly independent. If the restricted function $\tilde{f} := f|_\Gamma$ has a maximum or minimum value at x_0 then there exist constants $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$ such that

$$\nabla f(x_0) = \sum_{i=1}^{k} \lambda_i \nabla g_i(x_0)$$

We now have a general strategy for solving optimization problems.

1. Locate all critical points of f in Ω.
2. Determine if the boundary $\partial\Omega$ should be dealt with using basic geometry or Lagrange multipliers.
3. Use the method chosen in the previous step to identify all the optimal points on the boundary.
4. Evaluate f at every point found in steps 1 and 3, and the greatest value is the maximum of f while the smallest value is the minimum of f.

2 Inverse and Implicit Function Theorems

The inverse and implicit function theorems are powerful results in vector calculus, and their statements can take some time to digest. In fact, we use the inverse function theorem to prove the implicit function theorem.

Intuitively, the inverse function theorem generalizes the fact that for a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ with $f'(x_0) \neq 0$ then f is locally invertible (bijective) and we can calculate the derivative of its inverse in terms of f'. This should seem plausible: draw a picture of a function with non-zero derivative at some point x_0 and in a small neighbourhood of x_0 we should have f either strictly increasing or strictly decreasing, and so it is bijective since it is also continuous.

Theorem 5: Inverse Function Theorem

Let $f\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. Let $x_0 \in \Omega$ and assume $Df(x_0)$ is invertible. Then there exist neighbourhoods $U \ni x_0$ and $W \ni f(x_0)$ such that $f(U) = W$ and the restriction of f to U has a C^1 inverse map $f^{-1}\colon W \to U$. Furthermore, for all $y \in W$ and $x = f^{-1}(y)$ we have

$$Df^{-1}(y) = (Df(x))^{-1}$$

where $(Df(x))^{-1}$ is the inverse matrix of $Df(x)$. If $f \in C^k$ then $f^{-1} \in C^k$.

The implicit function tells us about solutions to systems of equations of the form

$$f(x_1, \dots, x_n) = 0$$

where f is at least continuously differentiable and satisfies some other assumptions at the chosen point x_0. In words, if the Jacobian of f at $x_0 = (x_0^1, \dots, x_0^n)$ is sufficiently structured, and $f(x_0^1, \dots, x_0^n) = 0$, then we can look at the equation $f(x_1, \dots, x_n) = 0$ for points near x_0 and actually rewrite it so that x_n is a C^1 function of the other $n-1$ variables.

Another way to say this is that locally (near x_0) the curve $f(x_1, \dots, x_n) = 0$ looks the graph of a function of $n-1$ variables. This is particularly natural when $n = 2$: it says that if you look near certain points on the curve $\{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$ then the curve looks like the graph of a C^1 function from $\mathbb{R}$ to $\mathbb{R}$. Without further ado let's look at the statement:

Theorem 6: Implicit Function Theorem

Let $F\colon \Omega \subseteq \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}^m$ be a C^k map, $k \geq 1$. Write $x \in \mathbb{R}^l \times \mathbb{R}^m$ as $x = (x', y)$ with $x \in \mathbb{R}^l$ and $y \in \mathbb{R}^m$ and $F = (F_1, \dots, F_m)$. Then for $x_0 = (x_0', y_0) \in \Omega$ with $F(x_0) = 0$, if the matrix

$$J_yF(x_0) := \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(x_0) & \cdots & \frac{\partial F_1}{\partial y_m}(x_0) \\ \vdots & \vdots & \vdots \\ \frac{\partial F_m}{\partial y_1}(x_0) & \cdots & \frac{\partial F_m}{\partial y_m}(x_0) \end{pmatrix}$$

is invertible then there exists a neighbourhood U of x_0' in $\mathbb{R}^l$ and a unique map $f\colon U \to \mathbb{R}^m$ such that

$$F(x', f(x')) = 0 \text{ and } f(x_0') = y_0$$

for all $x' \in U$. Moreover, $f \in C^k$ and we have

$$D_{x'}f(x') = -(J_yF(x))^{-1}D_{x'}F(x)$$

Let's look at a special case when $m = 1$ and $l = n - 1$. In other words we are taking a real-valued function and isolating the n-th coordinate of its argument.

Theorem 7: Implicit Function Theorem (Real-Valued)

Let $F\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be a C^k map, $k \geq 1$. Write $x' = (x_1, \ldots, x_{n-1})$. Suppose $F(x_0) = 0$ and $\dfrac{\partial F}{\partial x_n}(x_0) \neq 0$. Then there exists a neighbourhood U of x_0' in $\mathbb{R}^{n-1}$ and a unique map $f\colon U \to \mathbb{R}$ such that

$$F(x', f(x')) = 0 \text{ and } f(x_0') = y_0$$

for all $x' \in U$. Moreover, $f \in C^k$ and we have

$$\frac{\partial f}{\partial x_j}(x') = -\frac{\dfrac{\partial F}{\partial x_j}(x)}{\dfrac{\partial F}{\partial x_n}(x)}$$

Convince yourself that this implies for a function F satisfying these assumptions that near x_0 the level set $\{x \in \mathbb{R}^n \mid F(x) = 0\}$ can be expressed as the graph

$$\operatorname{gph}(f) := \{((x_1, \ldots, x_{n-1}), f(x_1, \ldots, x_{n-1})) \mid (x_1, \ldots, x_{n-1}) \in U\}$$

3 Exercises

1. (Marsden & Tromba 3.3 #43) Let $f(x, y) = 1 + xy - 2x + y$ and let D be the triangular region in $\mathbb{R}^2$ with vertices $(-2, 1)$, $(-2, 5)$ and $(2, 1)$. Find the absolute maximum and minimum values of f on D. Give all points where these extreme values occur.
2. (Marsden & Tromba 3.4 #15) Find the extrema of $f(x, y) = 4x + 2y$ subject to the constraint $2x^2 + 3y^2 = 21$
3. (Marsden & Tromba 3.4 #37) Find the point on the curve $(\cos t, \sin t, \sin(t/2))$ that is farthest from the origin.
4. Find the maximum and minimum values of $f(x, y, z) = y^2 - 10z$ subject to $x^2 + y^2 + z^2 = 36$.
5. (Marsden & Tromba 3.5 #1) Show that the equation $x + y - z + \cos(xyz) = 0$ can be solved for $z = g(x, y)$ near the origin. Find $\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial g}{\partial y}$ at $(0, 0)$.
6. (Marsden & Tromba 3.5 #7) Show that $x^3z^2 - z^3yx = 0$ is solvable for z as a function of (x, y) near $(1, 1, 1)$ but not near the origin. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at $(1, 1)$.
7. (Marsden & Tromba 3.5 #16) Consider the system of equations

$$x^5v^2 + 2y^3u = 3$$

$$3yu - xuv^3 = 2$$

Show that near the point $(x, y, u, v) = (1, 1, 1, 1)$ this system defines u and v implicitly as functions of x and y. For such local functions u and v define the local function $f(x, y) = (u(x, y), v(x, y))$. Find $Df(1, 1)$.