

MATH 248

Honours Vector Calculus

Tutorial Notes and Midterm Review

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1 Higher Order Derivatives

The Mean Value Theorem for functions of one variable is an extremely powerful and fundamental result in analysis, allowing us to understand behaviour of a function on an interval using the behaviour of its derivative. The result generalizes to real-valued functions defined on balls in $\mathbb{R}^n$.

Theorem 1: Mean Value Theorem

Let $B \subseteq \mathbb{R}^n$ be a ball, and suppose $f: B \rightarrow \mathbb{R}$ is differentiable on B . Then for all $x, y \in B$ there exists $z \in B$ such that

$$f(x) - f(y) = Df(z)(x - y)$$

where $Df(z)$ is the Jacobian of f at z . In particular,

$$|f(x) - f(y)| \leq \|Df(z)\| \|x - y\|$$

This is slightly restrictive since we require the range of f to be $\mathbb{R}$. If we allow the range of f to be $\mathbb{R}^m$ we are no longer guaranteed that the equality in Theorem 1 will hold, but we still have a useful inequality:

Theorem 2: Mean Value Inequality

Let $B \subseteq \mathbb{R}^n$ be a ball, and suppose $f: B \rightarrow \mathbb{R}^m$ is differentiable on B . Then for all $x, y \in B$ there exists $z \in B$ such that

$$\|f(x) - f(y)\| \leq \|Df(z)\| \|x - y\|$$

Recall that if $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable then it has partial derivatives $\frac{\partial f_i}{\partial x_j}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. We can then discuss the differentiability of these functions, which leads us to the concept of **higher order derivatives**.

Definition 1: C^k

Let $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ with Ω open. We say that $f \in C^k(\Omega)$ for an integer $k \geq 0$ if all of the partial derivatives of f up to order k exist and are continuous.

An important special case: $f: \Omega \rightarrow \mathbb{R}$, $f \in C^2(\Omega)$, we define the **Hessian** of f

at x to be:

$$H_f(x) = \begin{pmatrix} \frac{\partial^2 f_1}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f_1}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f_1}{\partial x_1 \partial x_n}(x) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f_1}{\partial x_n \partial x_1}(x) & \frac{\partial^2 f_1}{\partial x_n \partial x_2}(x) & \cdots & \frac{\partial^2 f_1}{\partial x_n \partial x_n}(x) \end{pmatrix}$$

Higher order partial derivatives can quickly become unwieldy if we have to consider all the possible orders in which they can be taken. Here we state a simple condition under which we can forget about the order and are guaranteed that the result will be the same no matter what.

Theorem 3: Equality of Mixed Partial

Suppose $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is twice differentiable. If $x \in \Omega$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}$, $\frac{\partial^2 f}{\partial x_j \partial x_i}$ are continuous at x then

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x)$$

In particular, if all of the second order partial derivatives of f are continuous at f then $H_f(x)$ is symmetric.

As a curiosity, the converse is not true. The function $f(x, y) = x^2 y^2 \sin(1/x) \sin(1/y)$ can be continuously extended to the lines $x = 0$ and $y = 0$, and it is an interesting exercise to show that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ while these functions are not continuous at $(0, 0)$. I found this example here: <https://math.stackexchange.com/questions/2095484/clairauts-theorem-is-reversible>

Taylor's theorem in one variable allows us to understand the local behaviour of a function near a point in terms of a polynomial with a remainder term. This can be generalized to higher dimensions, and the general formula is unwieldy but we will focus mainly on the first-order and second-order approximations (meaning in terms of first and second-order partial derivatives).

Theorem 4: Taylor's Theorem

Suppose $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is in $C^{k+1}(\Omega)$ and $x_0 \in \Omega$. Then

$$\begin{aligned} f(x) = & f(x_0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0)(x^i - x_0^i) \\ & + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)(x^i - x_0^i)(x^j - x_0^j) + \cdots \\ & + \frac{1}{k!} \sum_{\alpha_{i_1} + \cdots + \alpha_{i_k} = k} \frac{\partial^k f}{\partial x_{\alpha_{i_1}} \cdots \partial x_{\alpha_{i_k}}}(x_0)(x^{\alpha_{i_1}} - x_0^{\alpha_{i_1}}) \cdots (x^{\alpha_{i_k}} - x_0^{\alpha_{i_k}}) \\ & + R_{k+1}^f(x) \end{aligned}$$

where

$$R_{k+1}^f(x) = \frac{1}{(k+1)!} \sum_{\alpha_{i_1} + \cdots + \alpha_{i_{k+1}} = k+1} \frac{\partial^{k+1} f}{\partial x_{\alpha_{i_1}} \cdots \partial x_{\alpha_{i_{k+1}}}}(x_0)(x^{\alpha_{i_1}} - x_0^{\alpha_{i_1}}) \cdots (x^{\alpha_{i_{k+1}}} - x_0^{\alpha_{i_{k+1}}}) \quad (1)$$

which satisfies

$$\lim_{x \rightarrow x_0} \frac{R_{k+1}^f(x)}{\|x - x_0\|^k} = 0$$

The R_k^f term represents the error, and the condition at the bottom essentially tells us that the error goes to zero as we get closer to the point x_0 that we are expanding around.

The most important special cases are when $k = 0$ and $k = 1$:

$$f(x) = f(x_0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0)(x^i - x_0^i) + R_1^f(x)$$

$$f(x) = f(x_0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0)(x^i - x_0^i) + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)(x^i - x_0^i)(x^j - x_0^j) + R_2^f(x)$$

More compactly:

$$f(x) = f(x_0) + \nabla f(x_0)^T (x - x_0) + R_1^f(x)$$

$$f(x) = f(x_0) + \nabla f(x_0)^T (x - x_0) + \frac{1}{2} (x - x_0)^T H_f(x_0) (x - x_0) + R_2^f(x)$$

2 Optimization

An important application of derivatives is for optimizing real-valued functions. Many of the optimization theorems in one dimension have a corresponding result in higher dimensions, and we collect these results here. We will develop more sophisticated methods to solve constrained optimization problems in the coming lectures.

Definition 2: Local Extrema

Let $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$. We say that $x_0 \in \Omega$ is a **local maximum (minimum) point** if there exists $r > 0$ such that

$$f(x) \leq f(x_0) \quad (f(x) \geq f(x_0) \text{ respectively})$$

for all $x \in B_r(x_0)$.

Definition 3: Critical Points

Let $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$. We say that $x_0 \in \Omega$ is a **critical point** if f is differentiable at x_0 and $\nabla f(x_0) = 0$.

Theorem 5: First Derivative Test

Suppose $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at x_0 . If x_0 is a local maximum or minimum point of f then $\nabla f(x_0) = 0$.

Definition 4: Definite Matrix

A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called **positive semi-definite**, written $A \succeq 0$, if

$$x^T A x \geq 0$$

for all $x \in \mathbb{R}^n$. We say that A is **positive definite** if the inequality is strict for all non-zero $x \in \mathbb{R}^n$. We say that A is **negative (semi)-definite** if $-A \succeq 0$.

Intuitively, positive/negative (semi)-definite matrices are like non-negative/non-positive numbers (strictly non-zero in the definite case). The definiteness of the Hessian plays a role analogous to that of the sign of the second derivative when we are looking at critical points.

A symmetric matrix A is positive/negative (semi)-definite if and only if all the eigenvalues are non-negative/non-positive respectively, and also strictly non-zero in the definite case. This is a nice result but not useful for quick computations, so we point out two useful tricks for checking when small matrices are positive definite.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \succ 0 \iff \det(A) > 0 \text{ and } a > 0$$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \succ 0 \iff a > 0 \text{ and } \det \begin{pmatrix} a & b \\ d & e \end{pmatrix} > 0 \text{ and } \det(A) > 0$$

Theorem 6: Second Derivative Test

Suppose $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is C^3 in a neighbourhood of $x_0 \in \Omega$.

- a) If x_0 is a local maximum (minimum) point of f then $H_f(x_0)$ is negative (positive) semi-definite.
- b) If x_0 is a critical point of f and $H_f(x_0)$ is negative (positive) definite then x_0 is a local maximum (minimum) point of f .

3 Exercises

1. Show that the Mean Value Theorem can fail when the range is no longer one-dimensional. Hint: Consider the function $f(x) = (\cos x, \sin x)$.
2. (Marsden & Tromba 3.1 #32) Let

$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $(x, y) \neq (0, 0)$, and at the origin. Compute $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ at $(0, 0)$ and compare. Explain the result.

3. (Marsden & Tromba 3.1 #17) Suppose $f(x, y, z)$ is of class C^3 . Show that

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial^3 f}{\partial y \partial z \partial x}$$

4. (Marsden & Tromba 3.2 #11) Let $g(x, y) = \sin(xy) - 3x^2 \log y + 1$. Find the degree 2 polynomial which best approximates g near the point $(\pi/2, 1)$.
5. (Marsden & Tromba 3.3 #5) Find and classify the critical points of $f(x, y) = e^{1+x^2-y^2}$.
6. (Marsden & Tromba 3.3 #17) Find all local extrema for $f(x, y) = 8y^3 + 12x^2 - 24xy$.
7. (Marsden & Tromba 3.3 #18) Let $f(x, y, z) = x^2 + y^2 + z^2 + kyz$. Verify that $(0, 0, 0)$ is a critical point and determine all k such that $(0, 0, 0)$ is a local minimum.
8. (Marsden & Tromba 3.3 #46) We say $u \in C^2(\overline{B_1(0)})$ is strictly subharmonic if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} > 0$$

Show that u cannot have a maximum point in $B_1(0)$.

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1 Optimization

Last time we talked about critical points and their relationship with maxima and minima. In particular recall that the Second Derivative Test in higher dimensions tells us that positive/negative definiteness of the Hessian at a critical point guarantees us a minimum or maximum respectively, but the conclusion need not hold if we have only *semi*-definiteness. There is one more notable phenomenon that can occur at a critical point.

Definition 1: Saddle Point

Let $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ and $x_0 \in \Omega$ a critical point of f . If x_0 is neither a maximum nor a minimum we say that x_0 is a **saddle point**. In particular, for all $r > 0$ we can find points x_1 and x_2 in $B_r(x_0)$ with

$$f(x_1) < f(x_0) < f(x_2)$$

It is useful to know when a critical point is a saddle point. The following theorem gives us a sufficient condition.

Theorem 1: Saddle Point Existence

Let $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be C^3 and x_0 a critical point of f . If $\det(H_f(x_0)) \neq 0$ and $H_f(x_0)$ is neither positive definite nor negative definite then x_0 is a saddle point.

As we mentioned last time, checking if a general symmetric matrix is positive or negative definite can be challenging (for example, naively you could compute all of its eigenvalues). So to make our lives easier let's remember a trick for the case $n = 2$: If A is a 2×2 matrix and $\det(A) < 0$ then A is neither positive definite nor negative definite. We summarize the special case $n = 2$ in the following theorem:

Theorem 2: Second Derivative Test in $\mathbb{R}^2$

Let $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be C^2 and x_0 a critical point of f . Suppose $H_f(x_0) = \begin{pmatrix} f_{xx}(x_0) & f_{xy}(x_0) \\ f_{yx}(x_0) & f_{yy}(x_0) \end{pmatrix}$. Then

1. If $\det(H_f(x_0)) < 0$ then x_0 is a saddle point.
2. If $\det(H_f(x_0)) > 0$ and $f_{xx}(x_0) > 0$ then x_0 is a local minimum.
3. If $\det(H_f(x_0)) > 0$ and $f_{xx}(x_0) < 0$ then x_0 is a local maximum.
4. If $\det(H_f(x_0)) = 0$ the test is inconclusive.

We know from calculus that a continuous function on a closed and bounded set attains its extreme values. Our study of critical points allows us to deal with extreme points that lie “inside” of Ω (in its interior), but we need to be careful when searching for extreme points that lie on the boundary.

There are two main strategies for dealing with the function on the boundary.

1. If the boundary is sufficiently simple, for example a rectangle or triangle in $\mathbb{R}^2$ that is piecewise linear, then we can restrict our function to the lines defining the boundary and reduce it to a problem of 1-dimensional optimization.
2. If the boundary is given by the form $\{x \in \mathbb{R}^n \mid g(x) = 0\}$ for sufficiently nice g (i.e. $g \in C^1$) then we use the method of **Lagrange multipliers**

We will see examples of both approaches in the exercises. For now we study the theory behind Lagrange multipliers.

Theorem 3: Lagrange Multiplier Theorem

Let f and g be C^1 functions from $\mathbb{R}^n$ to $\mathbb{R}$. Suppose x_0 belongs to the level set

$$\Gamma = \{x \in \mathbb{R}^n \mid g(x) = 0\}$$

In other words, $g(x_0) = 0$. Suppose also that $Dg(x_0) \neq 0$. If the restricted function $\tilde{f} := f|_{\Gamma}$ has a maximum or minimum value at x_0 then there exists a constant $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_0) = \lambda \nabla g(x_0)$$

While the statement of this theorem is sophisticated, the key idea is that if we are solving an optimization problem of the form

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & g(x) = 0 \end{aligned} \tag{1}$$

then we are searching for optimizers of the function $\tilde{f}$ and the theorem tells us that we can try and study the system of equations

$$\nabla f(x) = \lambda \nabla g(x)$$

$$g(x) = 0$$

to solve our problem. There is a more general version of this theorem for multiple constraints. We state it here for completeness.

Theorem 4: General Lagrange Multiplier Theorem

Let f and g_i , $i = 1, \dots, k$ be C^1 functions from $\Omega \subseteq \mathbb{R}^n$ to $\mathbb{R}$. Let $x_0 \in \Omega$ and define

$$\Gamma = \{x \in \Omega \mid g_i(x) = g_i(x_0) \forall i = 1, \dots, k\}$$

Suppose also that $\nabla g_1(x_0), \dots, \nabla g_k(x_0)$ are linearly independent. If the restricted function $\tilde{f} := f|_{\Gamma}$ has a maximum or minimum value at x_0 then there exist constants $\lambda_i \in \mathbb{R}$, $i = 1, \dots, k$ such that

$$\nabla f(x_0) = \sum_{i=1}^k \lambda_i \nabla g_i(x_0)$$

We now have a general strategy for solving optimization problems.

1. Locate all critical points of f in Ω .
2. Determine if the boundary $\partial\Omega$ should be dealt with using basic geometry or Lagrange multipliers.
3. Use the method chosen in the previous step to identify all the optimal points on the boundary.
4. Evaluate f at every point found in steps 1 and 3, and the greatest value is the maximum of f while the smallest value is the minimum of f .

2 Inverse and Implicit Function Theorems

The inverse and implicit function theorems are powerful results in vector calculus, and their statements can take some time to digest. In fact, we use the inverse function theorem to prove the implicit function theorem.

Intuitively, the inverse function theorem generalizes the fact that for a differentiable function $f: \mathbb{R} \rightarrow \mathbb{R}$ with $f'(x_0) \neq 0$ then f is locally invertible (bijective) and we can calculate the derivative of its inverse in terms of f' . This should seem plausible: draw a picture of a function with non-zero derivative at some point x_0 and in a small neighbourhood of x_0 we should have f either strictly increasing or strictly decreasing, and so it is bijective since it is also continuous.

Theorem 5: Inverse Function Theorem

Let $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 map. Let $x_0 \in \Omega$ and assume $Df(x_0)$ is invertible. Then there exist neighbourhoods $U \ni x_0$ and $W \ni f(x_0)$ such that $f(U) = W$ and the restriction of f to U has a C^1 inverse map $f^{-1}: W \rightarrow U$. Furthermore, for all $y \in W$ and $x = f^{-1}(y)$ we have

$$Df^{-1}(y) = (Df(x))^{-1}$$

where $(Df(x))^{-1}$ is the inverse matrix of $Df(x)$. If $f \in C^k$ then $f^{-1} \in C^k$.

The implicit function tells us about solutions to systems of equations of the form

$$f(x_1, \dots, x_n) = 0$$

where f is at least continuously differentiable and satisfies some other assumptions at the chosen point x_0 . In words, if the Jacobian of f at $x_0 = (x_0^1, \dots, x_0^n)$ is sufficiently structured, and $f(x_0^1, \dots, x_0^n) = 0$, then we can look at the equation $f(x_1, \dots, x_n) = 0$ for points near x_0 and actually rewrite it so that x_n is a C^1 function of the other $n - 1$ variables.

Another way to say this is that locally (near x_0) the curve $f(x_1, \dots, x_n) = 0$ looks the graph of a function of $n - 1$ variables. This is particularly natural when $n = 2$: it says that if you look near certain points on the curve $\{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$ then the curve looks like the graph of a C^1 function from $\mathbb{R}$ to $\mathbb{R}$. Without further ado let's look at the statement:

Theorem 6: Implicit Function Theorem

Let $F: \Omega \subseteq \mathbb{R}^l \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a C^k map, $k \geq 1$. Write $x \in \mathbb{R}^l \times \mathbb{R}^m$ as $x = (x', y)$ with $x' \in \mathbb{R}^l$ and $y \in \mathbb{R}^m$ and $F = (F_1, \dots, F_m)$. Then for $x_0 = (x'_0, y_0) \in \Omega$ with $F(x_0) = 0$, if the matrix

$$J_y F(x_0) := \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(x_0) & \cdots & \frac{\partial F_1}{\partial y_m}(x_0) \\ \vdots & \vdots & \vdots \\ \frac{\partial F_m}{\partial y_1}(x_0) & \cdots & \frac{\partial F_m}{\partial y_m}(x_0) \end{pmatrix}$$

is invertible then there exists a neighbourhood U of x'_0 in $\mathbb{R}^l$ and a unique map $f: U \rightarrow \mathbb{R}^m$ such that

$$F(x', f(x')) = 0 \text{ and } f(x'_0) = y_0$$

for all $x' \in U$. Moreover, $f \in C^k$ and we have

$$D_{x'} f(x') = -(J_y F(x))^{-1} D_{x'} F(x)$$

Let's look at a special case when $m = 1$ and $l = n - 1$. In other words we are

taking a real-valued function and isolating the n -th coordinate of its argument.

Theorem 7: Implicit Function Theorem (Real-Valued)

Let $F: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^k map, $k \geq 1$. Write $x' = (x_1, \dots, x_{n-1})$. Suppose $F(x_0) = 0$ and $\frac{\partial F}{\partial x_n}(x_0) \neq 0$. Then there exists a neighbourhood U of x'_0 in $\mathbb{R}^{n-1}$ and a unique map $f: U \rightarrow \mathbb{R}$ such that

$$F(x', f(x')) = 0 \text{ and } f(x'_0) = y_0$$

for all $x' \in U$. Moreover, $f \in C^k$ and we have

$$\frac{\partial f}{\partial x_j}(x') = -\frac{\frac{\partial F}{\partial x_j}(x)}{\frac{\partial F}{\partial x_n}(x)}$$

Convince yourself that this implies for a function F satisfying these assumptions that near x_0 the level set $\{x \in \mathbb{R}^n \mid F(x) = 0\}$ can be expressed as the graph

$$\text{gph}(f) := \{((x_1, \dots, x_{n-1}), f(x_1, \dots, x_{n-1})) \mid (x_1, \dots, x_{n-1}) \in U\}$$

3 Exercises

- (Marsden & Tromba 3.3 #43) Let $f(x, y) = 1 + xy - 2x + y$ and let D be the triangular region in $\mathbb{R}^2$ with vertices $(-2, 1)$, $(-2, 5)$ and $(2, 1)$. Find the absolute maximum and minimum values of f on D . Give all points where these extreme values occur.
- (Marsden & Tromba 3.4 #15) Find the extrema of $f(x, y) = 4x + 2y$ subject to the constraint $2x^2 + 3y^2 = 21$
- (Marsden & Tromba 3.4 #37) Find the point on the curve $(\cos t, \sin t, \sin(t/2))$ that is farthest from the origin.
- Find the maximum and minimum values of $f(x, y, z) = y^2 - 10z$ subject to $x^2 + y^2 + z^2 = 36$.
- (Marsden & Tromba 3.5 #1) Show that the equation $x + y - z + \cos(xyz) = 0$ can be solved for $z = g(x, y)$ near the origin. Find $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ at $(0, 0)$.
- (Marsden & Tromba 3.5 #7) Show that $x^3z^2 - z^3yx = 0$ is solvable for z as a function of (x, y) near $(1, 1, 1)$ but not near the origin. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(1, 1)$.
- (Marsden & Tromba 3.5 #16) Consider the system of equations

$$x^5v^2 + 2y^3u = 3$$

$$3yu - xuv^3 = 2$$

Show that near the point $(x, y, u, v) = (1, 1, 1, 1)$ this system defines u and v implicitly as functions of x and y . For such local functions u and v define the local function $f(x, y) = (u(x, y), v(x, y))$. Find $Df(1, 1)$.

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1 Curves and Path Integrals

We will begin by studying functions from intervals in $\mathbb{R}$ to a higher dimensional space $\mathbb{R}^n$. Such functions have the geometric interpretation as curves or paths in $\mathbb{R}^n$, take for example the trajectory that a particle takes over a period of time.

Definition 1: C^1 Curve

Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$. We say that $\gamma \in C^1([a, b])$ if γ is C^1 on (a, b) and

$$\gamma'(a+) := \lim_{t \rightarrow a+} \frac{\gamma(t) - \gamma(a)}{t - a}$$

$$\gamma'(b-) := \lim_{t \rightarrow b-} \frac{\gamma(t) - \gamma(b)}{t - b}$$

exist and $\lim_{t \rightarrow a+} \gamma'(t) = \gamma'(a+)$, $\lim_{t \rightarrow b-} \gamma'(t) = \gamma'(b-)$.

An equivalent formulation is that γ is the restriction of a C^1 map from some open interval $(A, B) \supseteq (a, b)$ with $A < a, b < B$.

We can write any such γ as $\gamma(t) = (x_1(t), \dots, x_n(t))$ and taking the derivative gives us the **tangent vector**:

$$\gamma'(t) = (x'_1(t), \dots, x'_n(t))$$

Continuing our particle analogy, the tangent vector represents the velocity vector of the particle and its norm is the speed.

A natural problem is to measure the length of a curve, and using an argument based on polygonal paths it becomes clear that the proper definition is via an integral.

Definition 2: Length of a Curve

Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ belong to $C^1([a, b])$. We define the length element $ds = \|\gamma'(t)\|dt$. Then the **length** of the curve γ is

$$L(\gamma) := \int_a^b \|\gamma'(t)\|dt = \int_{\gamma} ds$$

For a C^1 function $f: [a, b] \rightarrow \mathbb{R}$ we can realize its graph as a curve γ defined by $\gamma(t) = (t, f(t))$. Then using the formula above we find

$$L(\gamma) = \int_a^b \sqrt{1 + (f'(t))^2} dt$$

We can enlarge our class of curves by ignoring issues of differentiability at finitely many points, as long as we still maintain continuity.

Definition 3: Curve Properties

Suppose $\gamma: [a, b] \rightarrow \mathbb{R}^n$ is continuous. We say γ is:

- a) **piecewise C^1** if there exists $a = a_0 < a_1 < \dots < a_m = b$ such that $\gamma \in C^1([a_{i-1}, a_i])$ for all $i = 1, \dots, m$.
- b) **regular** if $\gamma \in C^1([a, b])$ and $\gamma'(t) \neq 0$ for all $t \in [a, b]$.
- c) **piecewise smooth** if there exists $a = a_0 < a_1 < \dots < a_m = b$ such that γ is regular in each $[a_{i-1}, a_i]$ for all $i = 1, \dots, m$.

The next theorem gives us some intuition for why regular curves are nice. It tells us that regular curves look locally like graphs, with all coordinates dependent on only one of the coordinates.

Theorem 1: Regular Curve Property

If $\gamma: [a, b] \rightarrow \mathbb{R}^n$ is regular then there exists $a = t_0 < t_1 < \dots < t_N = b$ such that for all $i = 1, \dots, N$, on each $[t_{i-1}, t_i]$ there is an index j_i and a map $t \mapsto x_{j_i}(t)$ that is C^1 and invertible such that

$$\gamma(t) = (x_1(x_{j_i}(t)), \dots, x_{j_i-1}(x_{j_i}(t)), x_{j_i}(t), \dots, x_n(x_{j_i}(t)))$$

That is, γ is a graph over x_{j_i} .

Now we can generalize our definition of length to piecewise C^1 curves:

Definition 4: Length of a Piecewise C^1 Curve

Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ be piecewise C^1 on $[a, b]$. Then the **length** of the curve γ is

$$L(\gamma) := \sum_{i=1}^m \int_{a_{i-1}}^{a_i} \|\gamma'(t)\| dt$$

where the partition corresponds to the one given by the piecewise C^1 property.

Finally we can introduce a fundamental tool of vector calculus: the path integral (also called line integral).

Definition 5: Path Integral

Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ be piecewise C^1 on $[a, b]$. Suppose f is a continuous function defined along the image of γ . Then the **path integral** of f along γ is

$$\int_{\gamma} f ds := \int_a^b f(\gamma(t)) \|\gamma'(t)\| dt = \sum_{i=1}^m \int_{a_{i-1}}^{a_i} f(\gamma(t)) \|\gamma'(t)\| dt$$

where the partition corresponds to the one given by the piecewise C^1 property.

For the remainder of this section assume that $\gamma: [a, b] \rightarrow \mathbb{R}^3$ is a C^3 regular curve. Then its arc-length function

$$s(t) := \int_a^t \|\gamma'(u)\| du$$

is strictly increasing so the relationship between t and s is invertible. This allows us to write $t = h(s)$ for some function h . Then we can define $\tilde{\gamma}(s) := \gamma(h(s))$ and check that $\|\tilde{\gamma}'(s)\| = 1$ for all s . The function $\tilde{\gamma}(s)$ is called the arc-length parametrization of γ .

Now we know that we can parametrize by arc-length, let us assume going forward that we have already done so, i.e. γ is a function of s and satisfies $\|\gamma'(s)\| = 1$.

Then the tangent vector $T(s) = \gamma'(s)$ is a unit vector and $\gamma''(s) = T'(s)$ is orthogonal to $T(s)$.

There are a handful of related quantities and functions that we can associate with such curves. We collect them in the following definition.

Definition 6: Geometry of Curves

Suppose $\gamma: [a, b] \rightarrow \mathbb{R}^3$ is a C^3 regular curve parametrized by arc-length.

a) $\kappa(s) := \|T'(s)\|$ is the **curvature** of γ .

b) If $\kappa(s) > 0$ define the **principal normal vector** $N(s) = \frac{T'(s)}{\|T'(s)\|}$.

c) $B(s) := T(s) \times N(s)$ is the **binormal vector**.

It can be checked that $B'(s) = -\tau(s)N(s)$ and the function τ is called the **torsion** of γ .

2 Vector Fields

Vector fields are just functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. We will focus on differentiable vector fields and various operations we can do with them.

Definition 7: Vector Field

A function $F: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called a **vector field**. To every point $x \in \Omega$ it assigns another vector $F(x) \in \mathbb{R}^n$.

Take any differentiable function $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$. Then its gradient $\nabla f: \Omega \rightarrow \mathbb{R}^n$ is a vector field called the **gradient vector field**. We have to get through several definitions before we can do anything interesting.

Definition 8: Flow Lines

Let F be a vector field on $\Omega \subseteq \mathbb{R}^n$. A **flow line** for F is a curve γ mapping into Ω such that

$$\gamma'(t) = F(\gamma(t))$$

That is, F yields the velocity field of the path γ .

For any vector field F we can obtain a meaningful scalar function on Ω called the divergence of F .

Definition 9: Divergence

Let $F = (F_1, \dots, F_n)$ be a C^1 vector field on $\Omega \subseteq \mathbb{R}^n$. The **divergence** of F is the scalar function

$$\operatorname{div} F := \nabla \cdot F := \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}$$

Naturally we wonder if for any vector field F we can obtain a different meaningful vector field on Ω too. The problem is subtle, so we restrict our attention to $\mathbb{R}^3$ where the answer is affirmative and this field is called the curl of F .

Definition 10: Curl

Let $F = (F_1, F_2, F_3)$ be a C^1 vector field on $\Omega \subseteq \mathbb{R}^3$. The **curl** of F is the vector field

$$\operatorname{curl} F := \nabla \times F := \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right)$$

When you need to know the definition of curl, just remember the determinant formula for cross product. The curl is not a literal cross product, but you can think of it as a symbolic cross product that uses differentiation operators.

$$\operatorname{curl} F = \nabla \times F = \det \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{pmatrix} = \det \begin{pmatrix} i & \frac{\partial}{\partial x_1} & F_1 \\ j & \frac{\partial}{\partial x_2} & F_2 \\ k & \frac{\partial}{\partial x_3} & F_3 \end{pmatrix}$$

The physical significance of these quantities and fields will become more clear after we cover the fundamental theorems of calculus in $\mathbb{R}^2$ and $\mathbb{R}^3$.

We say that a vector field F is **incompressible** if $\operatorname{div} F = 0$, and that it is **irrotational** if $\operatorname{curl} F = 0$. There are various identities for the divergence and curl operators that you can prove. Some are in the lecture slides, and it is a good exercise of using the definitions to verify them.

We conclude with two special results for C^2 functions.

Theorem 2: Gradient is Irrotational

Let $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be C^2 . Then $\operatorname{curl} \nabla f = 0$.

Theorem 3: Curl is Incompressible

Let $F: \Omega \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a C^2 vector field. Then $\operatorname{div} \operatorname{curl} F = 0$.

3 Exercises

- (Marsden & Tromba 4.1 #11) Determine which of the following paths are regular.
 - $\gamma(t) = (\cos t, \sin t, t)$
 - $\gamma(t) = (t^3, t^5, \cos t)$
 - $\gamma(t) = (t^2, e^t, 3t + 1)$
- (Marsden & Tromba 4.2 #13) Let γ be the path $\gamma(t) = (2t, t^2, \log t)$ defined for $t > 0$. Find the arc-length of γ between the two points $(2, 1, 0)$ and $(4, 4, \log 2)$.
- (Marsden & Tromba 4.2 #14) Find the arc-length function for the curve $\gamma(t) = (\cosh t, \sinh t, t)$.
- (Marsden & Tromba 4.2 #18) Show that any line $l(t) = x_0 + tv$ has zero curvature, where v is a unit vector.
- (Straight lines are shortest) Let $\gamma: (a, b) \rightarrow \mathbb{R}^3$ be a C^1 curve. Let $[c, d] \subseteq (a, b)$, and set $\gamma(c) = p$, $\gamma(d) = q$.
 - Show that for any constant unit vector v

$$(q - p) \cdot v = \int_c^d \gamma'(t) \cdot v dt \leq \int_c^d \|\gamma'(t)\| dt$$

b) Set $v = \frac{q-p}{\|q-p\|}$ and show that

$$\|\gamma(d) - \gamma(c)\| \leq \int_c^d \|\gamma'(t)\| dt$$

that is, the curve of shortest length from $\gamma(c)$ and $\gamma(d)$ is the straight line joining these points.

6. (Marsden & Tromba 4.3 #18) Show that $\gamma(t) = (\sin t, \cos t, e^t)$ is a flow line for $F(x, y, z) = (y, -x, z)$.
7. (Marsden & Tromba 4.3 #21) Let $F(x, y, z) = (yz, xz, xy)$. Find a function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $F = \nabla f$. Do the same for $F(x, y, z) = (x, y, z)$.
8. (Marsden & Tromba 4.4 #3) Find the divergence of $F(x, y, z) = (x, y + \cos x, z + e^{xy})$.
9. (Marsden & Tromba 4.4 #14) Find the curl of $F(x, y, z) = (yz, xz, xy)$.
10. (Marsden & Tromba 4.4 #21) Let $F(x, y, z) = (x^2, x^2y, z + zx)$. Show that $\nabla \cdot (\nabla \times F) = 0$. Does there exist $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $F = \nabla f$?
11. (Marsden & Tromba 4.4 #28) Prove that $\operatorname{div}(\nabla f \times \nabla g) = 0$.

These notes were prepared by Ariel Goodwin for MATH 248 at McGill University as taught by Pengfei Guan.

1 Continuity and Differentiability

Perhaps the most fundamental definitions in this course, the ideas of continuity and differentiability are essential.

Definition 1: Continuity

A function $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **continuous** at $x_0 \in \Omega$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|f(x_0) - f(y)\| < \varepsilon$$

provided that $\|x_0 - y\| < \delta$. We say f is **continuous** on Ω if f is continuous at each $x \in \Omega$.

Definition 2: Differentiability

A function $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **differentiable** at $x_0 \in \Omega$ if there exists a linear map $L \in \mathbb{R}^{m \times n}$ such that

$$\lim_{x \rightarrow x_0} \frac{\|f(x) - f(x_0) - L(x - x_0)\|_{\mathbb{R}^m}}{\|x - x_0\|_{\mathbb{R}^n}} = 0$$

We say f is **differentiable** on Ω if f is differentiable at each $x \in \Omega$.

Theorem 1: Jacobian Properties

If $f = (f_1, \dots, f_m): \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at $x_0 \in \Omega$ then f is continuous at x_0 , all partial derivatives $\frac{\partial f_i}{\partial x_j}(x_0)$ exist for $i = 1, \dots, m$, $j = 1, \dots, n$, and the matrix

$$Df(x_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \cdots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) & \cdots & \frac{\partial f_m}{\partial x_n}(x_0) \end{pmatrix}$$

of partial derivatives of f at x_0 is the unique linear map L in the definition of the differentiation. This matrix $Df(x_0)$ is called the **Jacobian** or differential or derivative matrix.

Keep in mind that differentiability implies existence of partial derivatives, but existence of partial derivatives does not imply continuity.

Theorem 2: Continuity of Partial Derivatives is Sufficient

If $f = (f_1, \dots, f_m): \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ is such that all partial derivatives $\frac{\partial f_i}{\partial x_j}$ for $i = 1, \dots, m, j = 1, \dots, n$ exist and are continuous at x_0 , then f is differentiable at x_0 .

Do not forget that this condition is not necessary - there are functions that are differentiable at a point and their partial derivatives are not continuous at that point.

Theorem 3: Chain Rule

If $f: \Omega_1 \subseteq \mathbb{R}^n \rightarrow \Omega_2 \subseteq \mathbb{R}^m$ and $g: \Omega_2 \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^k$ are such that f is differentiable at $x \in \Omega_1$ and g is differentiable at $f(x) \in \Omega_2$ then $g \circ f$ is differentiable at x and

$$D(g \circ f)(x) = Dg(f(x))Df(x)$$

2 Mean Value Theorem, Taylor's Theorem

The Mean Value Theorem and its corresponding inequality are vital results that allow us to understand function behaviour in terms of derivative behaviour.

Theorem 4: Mean Value Theorem

Let $B \subseteq \mathbb{R}^n$ be a ball, and suppose $f: B \rightarrow \mathbb{R}$ is differentiable on B . Then for all $x, y \in B$ there exists $z \in B$ such that

$$f(x) - f(y) = Df(z)(x - y)$$

where $Df(z)$ is the Jacobian of f at z . In particular,

$$|f(x) - f(y)| \leq \|Df(z)\| \|x - y\|$$

Theorem 5: Mean Value Inequality

Let $B \subseteq \mathbb{R}^n$ be a ball, and suppose $f: B \rightarrow \mathbb{R}^m$ is differentiable on B . Then for all $x, y \in B$ there exists $z \in B$ such that

$$\|f(x) - f(y)\| \leq \|Df(z)\| \|x - y\|$$

Two important theorems on higher-order derivatives: equality of mixed partials allows us to take partial derivatives in whatever order we please, and Taylor's theorem allows us to approximate functions using polynomials with an explicit form for the error.

Theorem 6: Equality of Mixed Partial

Suppose $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is twice differentiable. If $x \in \Omega$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}$, $\frac{\partial^2 f}{\partial x_j \partial x_i}$ are continuous at x then

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x)$$

In particular, if all of the second order partial derivatives of f are continuous at f then the Hessian $H_f(x)$ is symmetric.

Theorem 7: Taylor's Theorem

Suppose $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is in $C^{k+1}(\Omega)$ and $x_0 \in \Omega$. Then

$$\begin{aligned} f(x) &= f(x_0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0)(x^i - x_0^i) \\ &+ \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)(x^i - x_0^i)(x^j - x_0^j) + \cdots \\ &+ \frac{1}{k!} \sum_{\alpha_{i_1} + \cdots + \alpha_{i_k} = k} \frac{\partial^k f}{\partial x_{\alpha_{i_1}} \partial x_{\alpha_{i_k}}}(x_0)(x^{\alpha_{i_1}} - x_0^{\alpha_{i_1}}) \cdots (x^{\alpha_{i_k}} - x_0^{\alpha_{i_k}}) \\ &+ R_{k+1}^f(x) \end{aligned}$$

where

$$R_{k+1}^f(x) = \frac{1}{(k+1)!} \sum_{\alpha_{i_1} + \cdots + \alpha_{i_{k+1}} = k+1} \frac{\partial^{k+1} f}{\partial x_{\alpha_{i_1}} \cdots \partial x_{\alpha_{i_{k+1}}}}(x_0)(x^{\alpha_{i_1}} - x_0^{\alpha_{i_1}}) \cdots (x^{\alpha_{i_{k+1}}} - x_0^{\alpha_{i_{k+1}}}) \quad (1)$$

which satisfies

$$\lim_{x \rightarrow x_0} \frac{R_{k+1}^f(x)}{\|x - x_0\|^k} = 0$$

3 Optimization

The theory of critical points and the derivative tests to find and classify critical points is the cornerstone of smooth optimization.

Theorem 8: First Derivative Test

Suppose $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at x_0 . If x_0 is a local maximum or minimum point of f then $\nabla f(x_0) = 0$.

Theorem 9: Second Derivative Test

Suppose $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is C^3 in a neighbourhood of $x_0 \in \Omega$.

- a) If x_0 is a local maximum (minimum) point of f then $H_f(x_0)$ is negative (positive) semi-definite.
- b) If x_0 is a critical point of f and $H_f(x_0)$ is negative (positive) definite then x_0 is a local maximum (minimum) point of f .

Don't forget that saddle points can also occur at critical points.

Definition 3: Saddle Point

Let $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ and $x_0 \in \Omega$ a critical point of f . If x_0 is neither a maximum nor a minimum we say that x_0 is a **saddle point**. In particular, for all $r > 0$ we can find points x_1 and x_2 in $B_r(x_0)$ with

$$f(x_1) < f(x_0) < f(x_2)$$

Theorem 10: Saddle Point Existence

Let $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be C^3 and x_0 a critical point of f . If $\det(H_f(x_0)) \neq 0$ and $H_f(x_0)$ is neither positive definite nor negative definite then x_0 is a saddle point.

All of the techniques above deal with optimization in open sets. To handle optimization on compact sets that include their boundary, we need some more machinery. There are two main strategies for optimizing the function on the boundary.

1. If the boundary is sufficiently simple, for example a rectangle or triangle in $\mathbb{R}^2$ that is piecewise linear, then we can restrict our function to the lines defining the boundary and reduce it to a problem of 1-dimensional optimization.
2. If the boundary is given by the form $\{x \in \mathbb{R}^n \mid g(x) = 0\}$ for sufficiently nice g (i.e. $g \in C^1$) then we use the method of **Lagrange multipliers**.

Theorem 11: Lagrange Multiplier Theorem

Let f and g be C^1 functions from $\mathbb{R}^n$ to $\mathbb{R}$. Suppose x_0 belongs to the level set

$$\Gamma = \{x \in \mathbb{R}^n \mid g(x) = 0\}$$

In other words, $g(x_0) = 0$. Suppose also that $Dg(x_0) \neq 0$. If the restricted function $\tilde{f} := f|_{\Gamma}$ has a maximum or minimum value at x_0 then there exists a constant $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_0) = \lambda \nabla g(x_0)$$

While the statement of this theorem is sophisticated, the key idea is that if we are solving an optimization problem of the form

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & g(x) = 0 \end{aligned} \tag{2}$$

then we are searching for optimizers of the function $\tilde{f}$ and the theorem tells us that we can try and study the system of equations

$$\nabla f(x) = \lambda \nabla g(x)$$

$$g(x) = 0$$

to solve our problem. The more general version of this theorem for multiple constraints is given below.

Theorem 12: General Lagrange Multiplier Theorem

Let f and $g_i, i = 1, \dots, k$ be C^1 functions from $\Omega \subseteq \mathbb{R}^n$ to $\mathbb{R}$. Let $x_0 \in \Omega$ and define

$$\Gamma = \{x \in \Omega \mid g_i(x) = g_i(x_0) \forall i = 1, \dots, k\}$$

Suppose also that $\nabla g_1(x_0), \dots, \nabla g_k(x_0)$ are linearly independent. If the restricted function $\tilde{f} := f|_{\Gamma}$ has a maximum or minimum value at x_0 then there exist constants $\lambda_i \in \mathbb{R}, i = 1, \dots, k$ such that

$$\nabla f(x_0) = \sum_{i=1}^k \lambda_i \nabla g_i(x_0)$$

We now have a general strategy for solving optimization problems.

1. Locate all critical points of f in Ω .
2. Determine if the boundary $\partial\Omega$ should be dealt with using basic geometry or Lagrange multipliers.
3. Use the method chosen in the previous step to identify all the optimal points on the boundary.
4. Evaluate f at every point found in steps 1 and 3, and the greatest value is the maximum of f while the smallest value is the minimum of f .

4 Exercises

1. Define $f(x, y) = \frac{x^6 y^2}{x^8 + y^4}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Determine the differentiability of f at the origin and the continuity of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ at the origin.
2. Suppose $f: \mathbb{R}^n \rightarrow \mathbb{R}, g: \mathbb{R}^n \rightarrow \mathbb{R}$ are differentiable. Show that the product function $h(x) = f(x)g(x)$ is differentiable and that

$$Dh(x_0) = f(x_0)Dg(x_0) + g(x_0)Df(x_0)$$

3. Suppose $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable, and satisfies

$$f(x) = 0 \quad \forall x : \|x\| = 1$$

$$\|\nabla f(x)\| \leq 1 \quad \forall x \in B_1(0)$$

Show that $|f(x)| \leq 1$ on $\overline{B_1(0)}$.

4. Find the shortest distance from the point $(3, 2, 1, \dots, 1) \in \mathbb{R}^n$ ($n \geq 3$) to the hyperplane with equation $x_1 + x_2 + \dots + x_n = 1$.
5. Determine the nature of the critical points of $f(x, y) = x^3 + y^2 - 6xy + 6x + 3y$.
6. Calculate the second-order Taylor approximation to $f(x, y) = \cos x \sin y$ at $(\pi, \pi/2)$.
7. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^2 function with

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = 0$$

for all $i, j = 1, \dots, n$ and $x \in \mathbb{R}^n$. Show that there exist constants $a_0, a_1, \dots, a_n \in \mathbb{R}$ such that

$$f(x) = a_0 + \sum_{i=1}^n a_i x_i$$

for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. What are the values of these constants?

8. Design a cylindrical can (with lid) to contain 1L of water using the minimum amount of metal.

These notes were prepared by Ariel Goodwin for MATH 248 at McGill University as taught by Pengfei Guan.

1 The Riemann Integral

In one dimension, the integral of a non-negative real-valued function can be interpreted as the area between the graph and the x -axis. In two dimensions, the integral of a non-negative real-valued function can be interpreted as the volume between the graph and the xy -plane. We need some technical definitions to make sense of these notions rigorously.

Definition 1: Partition

Let $R = [a, b] \times [c, d]$ be a rectangle in $\mathbb{R}^2$. A **partition** of R is two ordered sets

$$\mathcal{P} = \{x_0, x_1, \dots, x_n; y_0, y_1, \dots, y_m\}$$

such that

$$a = x_0 < x_1 < \dots < x_n = b, c = y_0 < y_1 < \dots < y_m = d$$

If we have two partitions,

$$\mathcal{P} = \{x_0, x_1, \dots, x_n; y_0, y_1, \dots, y_m\}$$

$$\tilde{\mathcal{P}} = \{\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_k; \tilde{y}_0, \tilde{y}_1, \dots, \tilde{y}_l\}$$

we say that $\tilde{\mathcal{P}}$ is a **refinement** of $\mathcal{P}$ if

$$\{x_0, x_1, \dots, x_n\} \subseteq \{\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_k\}$$

$$\{y_0, y_1, \dots, y_m\} \subseteq \{\tilde{y}_0, \tilde{y}_1, \dots, \tilde{y}_l\}$$

We also denote by $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$. These are the sub-rectangles that make up the partition. Throughout this section R will denote the rectangle $[a, b] \times [c, d]$.

Definition 2: Lower and Upper Estimates

Let $f: R \rightarrow \mathbb{R}$ be bounded. Then for any partition $\mathcal{P}$ of R , define

$$L(f, \mathcal{P}) = \sum_{j=1}^m \sum_{i=1}^n \inf_{x \in R_{ij}} f(x) \Delta x_i \Delta y_j$$

$$U(f, \mathcal{P}) = \sum_{j=1}^m \sum_{i=1}^n \sup_{x \in R_{ij}} f(x) \Delta x_i \Delta y_j$$

The next result essentially says that as you take your partitions finer and finer, your lower and upper estimates of the integral will get better and approach each

other. However, they may not approach each other entirely, and such functions are not Riemann-integrable by definition.

Theorem 1: Refinement Bounds

Let $f: R \rightarrow \mathbb{R}$ be bounded, and let $\mathcal{P}, \tilde{\mathcal{P}}$ be partitions of R such that $\tilde{\mathcal{P}}$ is a refinement of $\mathcal{P}$. Then

$$L(f, \mathcal{P}) \leq L(f, \tilde{\mathcal{P}}) \leq U(f, \tilde{\mathcal{P}}) \leq U(f, \mathcal{P})$$

Finally we can define the Riemann integral for functions whose lower and upper estimates do converge towards each other with equality in the limit.

Definition 3: Riemann Integral

Let $f: R \rightarrow \mathbb{R}$ be bounded. Define

$$\overline{\iint_R} f(x) dA = \inf \{U(f, \mathcal{P}) \mid \mathcal{P} \text{ partition of } R\}$$

$$\underline{\iint_R} f(x) dA = \sup \{L(f, \mathcal{P}) \mid \mathcal{P} \text{ partition of } R\}$$

We say f is **Riemann-integrable** over R if $\overline{\iint_R} f(x) dA = \underline{\iint_R} f(x) dA$ in which case we just write

$$\iint_R f(x) dA = \overline{\iint_R} f(x) dA = \underline{\iint_R} f(x) dA$$

What functions are Riemann-integrable? It turns out that continuous functions and functions that are continuous everywhere except for some well-behaved subsets are Riemann-integrable.

Theorem 2: Riemann-Integrable Functions

Let $f: R \rightarrow \mathbb{R}$ be bounded. If f is continuous, then f is Riemann-integrable. More generally, if $\gamma_i \subseteq R, i = 1, \dots, k$ are the curves of continuous functions (either over the x -axis or y -axis) such that f is continuous on $R \setminus \bigcup_{i=1}^k \gamma_i$ then f is Riemann-integrable.

The integral satisfies the following familiar properties.

Theorem 3: Integral Properties

Suppose f, g are Riemann-integrable over R . Then

a) $f + g$ is Riemann-integrable over R and

$$\iint_R (f + g) dA = \iint_R f dA + \iint_R g dA$$

b) For all $c \in \mathbb{R}$, cf is Riemann-integrable over R and

$$\iint_R cf dA = c \iint_R f dA$$

c) If $f(x, y) \geq g(x, y)$ on R then

$$\iint_R f dA \geq \iint_R g dA$$

d) If $R = R_1 \cup \dots \cup R_k$ with $R_i, i = 1, \dots, k$ pairwise disjoint rectangles then

$$\iint_R f dA = \sum_{i=1}^k \iint_{R_i} f dA$$

Exercise 1. If $f(x, y) = e^{\sin(x+y)}$ on $D = [-\pi, \pi] \times [-\pi, \pi]$, show that

$$\frac{1}{e} \leq \frac{1}{4\pi^2} \iint_D f(x, y) dA \leq e$$

Now we wish to extend the integral to functions defined on more general regions than rectangles.

Definition 4: Integral on Bounded Regions

Let $D \subseteq \mathbb{R}^2$ be a bounded region and $f: D \rightarrow \mathbb{R}$ be bounded. Let $R \supseteq D$ be a rectangle. Define a function on R :

$$f_R^*(x, y) = \begin{cases} f(x, y) & (x, y) \in D \\ 0 & (x, y) \notin D \end{cases}$$

We say f is Riemann-integrable over D if there is a rectangle $R \supseteq D$ such that f_R^* is Riemann-integrable on R and we define

$$\iint_D f dA = \iint_R f_R^* dA$$

Exercise 2. Let f be continuous, $f \geq 0$ on R . Show that if $\iint_R f dA = 0$ then $f = 0$ on R .

At this point we define some nice regions in the plane that we can often compute integrals over without too much trouble.

Definition 5: Simple Regions

Let $D \subseteq \mathbb{R}^2$. Then D is called a **y -simple region** if there exist continuous functions $\phi_1, \phi_2: [a, b] \rightarrow \mathbb{R}$ such that

$$D = \{(x, y) \mid a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)\}$$

Similarly, D is called an **x -simple region** if there exist continuous functions $\psi_1, \psi_2: [c, d] \rightarrow \mathbb{R}$ such that

$$D = \{(x, y) \mid c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y)\}$$

D is called a **simple region** if it is both x -simple and y -simple. Any of these regions is called an **elementary region**.

Theorem 4: Integral on Elementary Regions

Let $D = \bigcup_{i=1}^k D_i$ with each D_i , $i = 1, \dots, k$ an elementary region. Suppose $\text{int } D_i \cap \text{int } D_j = \emptyset$ for all $i \neq j$. Suppose $\gamma_i \subseteq D$, $i = 1, \dots, m$ are curves of continuous functions (either over the x -axis or y -axis) and suppose that f is continuous on $D \setminus \bigcup_{i=1}^m \gamma_i$. Then f is integrable on D .

Double integrals are nice in theory but computing them from the definition is difficult to do by hand. We can use Fubini's theorem to write many double integrals as iterated integrals, and then use our knowledge of one-dimensional calculus (e.g. FTC) to carry out the integration.

Theorem 5: Fubini's Theorem

Let $f: R \rightarrow \mathbb{R}$ be Riemann-integrable over R and suppose for each $x \in [a, b]$ that $\int_c^d f(x, y) dy$ exists. Then $\int_a^b \int_c^d f(x, y) dy dx$ exists and

$$\int_a^b \int_c^d f(x, y) dy dx = \iint_R f dA$$

Exercise 3. Compute the volume of the solid bounded by the surface $z = \sin y$, the planes $x = 1, x = 0, y = 0, y = \pi/2$, and the xy -plane.

Of course, the symmetric statement interchanging x and y is also true. We can cover all of these cases in the following general theorem:

Theorem 6: Iterated Integrals on Elementary Regions

Let $f: D \rightarrow \mathbb{R}$ where f, D satisfy the assumptions of Theorem 4. Then

- a) If D is y -simple and for every $x \in [a, b]$, $\int_{\phi_1(x)}^{\phi_2(x)} f(x, y) dy$ exists, then $\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) dy dx$ exists and

$$\iint_D f dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) dy dx$$

- b) If D is x -simple and for every $y \in [c, d]$, $\int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx$ exists, then $\int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx dy$ exists and

$$\iint_D f dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx dy$$

Exercise 4. Show that $\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dy dx = \frac{\pi}{4}$ but that the iterated integral in the opposite order is equal to $-\frac{\pi}{4}$. Does this contradict Fubini's theorem?

Exercise 5. Evaluate $\int_0^1 \int_{\sqrt{y}}^1 e^{x^3} dx dy$.

Exercise 6. Evaluate $\iint_D x dA$ where D is the interior of the triangle with vertices $(-3, 0), (6, 0), (0, 3)$.

We also have a mean value theorem for double integrals.

Theorem 7: Mean Value Theorem for Double Integrals

Suppose $f: D \rightarrow \mathbb{R}$ is continuous on an elementary region $D \subseteq \mathbb{R}^2$. Then there is a point $(x_0, y_0) \in D$ such that

$$\iint_D f dA = f(x_0, y_0) A(D)$$

where $A(D)$ is the area of D defined by

$$A(D) = \iint_D 1 dA$$

Let us introduce some notation briefly: Suppose $x = x(u, v), y = y(u, v)$ are C^1 functions of u, v . Then we write

$$\frac{\partial(x, y)}{\partial(u, v)} := \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

The change of variables theorem allows us to transform the integrand and domain of integration simultaneously to obtain a simpler equivalent integral.

Theorem 8: Change of Variables Theorem

Let D, D^* be elementary regions in $\mathbb{R}^2$, and let $T: D^* \rightarrow D$ be a bijective C^1 map given by $x = x(u, v), y = y(u, v)$. Then for any integrable function $f: D \rightarrow \mathbb{R}$:

$$\iint_D f(x, y) dx dy = \iint_{D^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

Exercise 7. Calculate $\iint_R (x + y)^2 e^{x-y} dA$ where R is the region bounded by $x - y = -1, x - y = 1, x + y = 1, x + y = 4$.

These notes were prepared by Ariel Goodwin for MATH 248 at McGill University as taught by Pengfei Guan.

1 Triple Integrals

The development and definitions for triple integrals are completely analogous to the double integral. The key idea is that partitions are now collections of three ordered sets, the rectangles R_{ijk} defined by these partitions are three-dimensional boxes, the function f is defined on a rectangular region $W \subseteq \mathbb{R}^3$, and the notation now has three integrals rather than two.

Many of the theorems from last Friday's tutorial remain true with slight modifications to notation. The refinement bounds in Theorem 1 remain true, as do the integral properties in Theorem 3. In particular, triple integrals are still linear in the sense that

$$\iiint_W (af + bg) dV = a \iiint_W f dV + b \iiint_W g dV$$

and they are monotone in the sense that if $f \geq g$ on W then

$$\iiint_W f dV \geq \iiint_W g dV$$

We take a minute to record a useful lemma on checking if a function is Riemann-integrable. It is also valid for the double integrals we discussed last time (just replace W by R).

Theorem 1: Integrability Criterion

Let $f: W \rightarrow \mathbb{R}$ be bounded. Then f is Riemann-integrable over W if and only if for all $\varepsilon > 0$ there exists a partition $\mathcal{P}$ such that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$$

We also mention the analogue of Theorem 2:

Theorem 2: Riemann-integrable Functions

Let $f: W \rightarrow \mathbb{R}$ be bounded. If f is continuous then f is Riemann-integrable over W .

More generally, if $\Gamma_i \subseteq W$, $i = 1, \dots, k$ are graphs of continuous real-valued functions on bounded regions in $\mathbb{R}^2$ (i.e., surfaces) and f is continuous on $W \setminus \bigcup_{i=1}^k \Gamma_i$ then f is Riemann-integrable over W .

The extension of the integral to non-rectangular regions follows the same idea as the double integral. For a general bounded region W , find a rectangular region $\tilde{W} \supseteq W$ and define an auxiliary function $\tilde{f}$ that agrees with f on W and is 0 otherwise. Then the integral of f on W is defined to be the integral of $\tilde{f}$ on $\tilde{W}$ (if it exists).

Simple regions in $\mathbb{R}^3$ are also defined in a natural way that reminds us of the two-dimensional case.

Definition 1: Simple Regions

Let $W \subseteq \mathbb{R}^3$. Then W is called a **z -simple region** if there is an elementary region $D \subseteq \mathbb{R}^2$ and continuous functions $\eta_1, \eta_2: D \rightarrow \mathbb{R}$ such that

$$W = \{(x, y, z) \mid \eta_1(x, y) \leq z \leq \eta_2(x, y) \forall (x, y) \in D\}$$

Similarly, **x -simple** and **y -simple** regions are defined by changing the coordinates in the above definition appropriately. D is called a **simple region** if it is x -simple, y -simple, and z -simple. Any of these regions is called an **elementary region**.

Now we can introduce Fubini's theorem for triple integrals. Unsurprisingly, it looks like Fubini's theorem for double integrals.

Theorem 3: Fubini's Theorem

Let $f: W \rightarrow \mathbb{R}$ with W bounded and piecewise-smooth (its boundary is a union of finitely many graphs of smooth functions). Suppose the conditions of Theorem 2 hold. Then f is integrable on W and if furthermore W is z -simple and $\int_{\eta_1(x,y)}^{\eta_2(x,y)} f(x, y, z) dz$ exists for all $(x, y) \in D$ then

$$\iiint_W f dV = \iint_D \left(\int_{\eta_1(x,y)}^{\eta_2(x,y)} f(x, y, z) dz \right) dA$$

Analogous formulas hold if W is x -simple or y -simple.

Exercise 1. Evaluate $\iiint_W x^2 \cos z dx dy dz$ where W is the region bounded by the planes $z = 0, z = \pi, y = 0, y = 1, x = 0, x + y = 1$.

Exercise 2. Evaluate $\iiint_W (1 - z^2) dx dy dz$ where W is the pyramid with top vertex at $(0, 0, 1)$, and base vertices at $(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0)$.

Exercise 3. Find the appropriate limits $\phi_1(x), \phi_2(x), \eta_1(x, y), \eta_2(x, y)$ to write

$$\iiint_W f dV = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\eta_1(x,y)}^{\eta_2(x,y)} f(x, y, z) dz dy dx$$

where $W = \{(x, y, z) \mid x^2 + y^2 \leq 1, z \geq 0, x^2 + y^2 + z^2 \leq 4\}$.

Let us introduce some notation briefly: Suppose $x = x(u, v, w), y = y(u, v, w), z = z(u, v, w)$ are C^1 functions of u, v, w . Then we write

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} := \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

Theorem 4: Change of Variables Theorem

Let W, W^* be bounded piecewise-smooth regions in $\mathbb{R}^3$, and let $T: W^* \rightarrow W$ be a bijective C^1 map given by $x = x(u, v, w), y = y(u, v, w), z = z(u, v, w)$. Then for any integrable function $f: W \rightarrow \mathbb{R}$:

$$\iiint_W f(x, y, z) dx dy dz = \iiint_{W^*} f(u, v, w) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$$

Exercise 4. Evaluate $\iiint_W \frac{dx dy dz}{(x^2 + y^2 + z^2)^{3/2}}$ where W is the region bounded by the two spheres $x^2 + y^2 + z^2 = a^2, x^2 + y^2 + z^2 = b^2$ with $0 < b < a$.

2 Surface Integrals

A surface S is the graph of a C^1 function $g: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ with D piecewise smooth and bounded:

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid z = g(x, y), (x, y) \in D\}$$

Geometric arguments from class lead us to natural definition of area for such surfaces

$$A(S) = \iint_D \sqrt{1 + g_x^2 + g_y^2} dx dy$$

We can be much more general than this, and we encapsulate this generality in the following definition.

Definition 2: Parametrized Surface

A **parametrization of a surface** is a map $\Phi: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$, and the **parametrized surface** S is the image $\Phi(D)$:

$$S = \{(x(u, v), y(u, v), z(u, v)) = \Phi(u, v) \mid (u, v) \in D\}$$

We say S is a C^1 surface if Φ is C^1 , and S is a **regular** surface if $\Phi_u \times \Phi_v \neq 0$ for all $(u, v) \in D$.

Again, geometric considerations lead us to define the tangent vectors $T_u = \Phi_u$, $T_v = \Phi_v$, a normal vector to the surface $T_u \times T_v$, and the area element of the surface S as $dS = \|T_u \times T_v\|$, from which we define

$$A(S) = \iint_D \|T_u \times T_v\| dA$$

It is a good idea to verify that we can recover the special case when S is a graph using this formula.

Exercise 5. Find an expression for a unit vector normal to the surface

$$x = \cos v \sin u, y = \sin v \sin u, z = \cos u$$

at the image of a point $(u, v) \in [0, \pi] \times [0, 2\pi]$. What is this surface?

Exercise 6. Find the area of the surface defined by $x + y + z = 1, x^2 + 2y^2 \leq 1$.

Now we can introduce another key construct of vector calculus - the surface integral.

Definition 3: Surface Integral

Let S be a parametrized surface and $f: S \rightarrow \mathbb{R}$ a bounded function. We define the **surface integral of f on S** to be

$$\iint_S f dS := \iint_D f(\Phi(u, v)) \|T_u \times T_v\| du dv$$

Exercise 7. Let S be the surface defined by $\Phi(u, v) = (u + v, u - v, uv)$. Show that the image of Φ is in the graph of the surface $4z = x^2 - y^2$. Then evaluate $\iint_S x dS$ for all points on the graph S over $x^2 + y^2 \leq 1$.

These notes were prepared by Ariel Goodwin for MATH 248 at McGill University as taught by Pengfei Guan.

1 Line Integrals

The definition of the line integral is motivated by calculating the work done by a force field F acting on a particle that moves through a path γ .

Definition 1: Line Integral

Let $\gamma: [a, b] \rightarrow \mathbb{R}^3$ be a piecewise-smooth path and F a vector field defined in a neighbourhood of the curve $\gamma([a, b]) \subseteq \mathbb{R}^3$. The **line integral** of F along γ is

$$\int_{\gamma} F \cdot ds := \int_a^b F(\gamma(t)) \cdot \gamma'(t) dt$$

Exercise 1. Compute the path integral of $f(x, y, z) = xyz$ on $\gamma(t) = (\cos t, \sin t, t)$ for $0 \leq t \leq \pi/2$.

Exercise 2. Evaluate $\int_{\gamma} ydx + (3y^3 - x)dy + zdz$ for all curves in the family $\gamma(t) = (t, t^n, 0)$, $n = 1, 2, \dots$, $0 \leq t \leq 1$.

Exercise 3. Let $\gamma: [a, b] \rightarrow \mathbb{R}^3$ be a path, and T its unit tangent vector. What is $\int_{\gamma} T \cdot ds$?

Curves can be reparametrized according to the following definition. A reparametrization changes how the curve is traversed as time evolves (e.g. twice as fast), but the resultant image in $\mathbb{R}^3$ is preserved.

Definition 2: Reparametrization

Suppose $\gamma: [a, b] \rightarrow \mathbb{R}^3$ is a piecewise-smooth path. A path $\tilde{\gamma}$ is called a **reparametrization** of γ if there is a bijective C^1 map $h: [a', b'] \rightarrow [a, b]$ such that

$$\tilde{\gamma}(t) = \gamma(h(t)) \quad \forall t \in [a', b']$$

A reparametrization is called **orientation-preserving** if

$$\tilde{\gamma}(a') = \gamma(a), \quad \tilde{\gamma}(b') = \gamma(b)$$

A reparametrization is called **orientation-reversing** if

$$\tilde{\gamma}(a') = \gamma(b), \quad \tilde{\gamma}(b') = \gamma(a)$$

Theorem 1: Orientation of Line Integrals

Suppose $\gamma: [a, b] \rightarrow \mathbb{R}^3$ is a piecewise-smooth path, and F is a continuous vector field on γ . Let $\tilde{\gamma}$ be a reparametrization of γ . If $\tilde{\gamma}$ is orientation-preserving then

$$\int_{\tilde{\gamma}} F \cdot ds = \int_{\gamma} F \cdot ds$$

If $\tilde{\gamma}$ is orientation-reversing then

$$\int_{\tilde{\gamma}} F \cdot ds = - \int_{\gamma} F \cdot ds$$

Note that path integrals of scalar functions are independent of any reparametrization. Now we get a version of the fundamental theorem of calculus for line integrals.

Theorem 2: FTC for Line Integrals

Suppose $F = \nabla f$ for some C^1 function f defined in a neighbourhood of the piecewise-smooth path γ . Then

$$\int_{\gamma} F \cdot ds = f(\gamma(b)) - f(\gamma(a))$$

The condition that F is a gradient of some smooth function is useful, so we give it a name.

Definition 3: Conservative Vector Field

Suppose F is a vector field on $\Omega \subseteq \mathbb{R}^3$. We say F is a **conservative** vector field in Ω if there exists a C^1 function $f: \Omega \rightarrow \mathbb{R}$ such that $F(x) = \nabla f(x)$ for all $x \in \Omega$.

You might recall from physics courses that work done by conservative forces is independent of the path taken by an object under the influence of the force. This definition and the previous theorem are the proper mathematical formulation of those ideas.

Definition 4: Simple Curves

A curve C is called a **simple curve** if C is the image of a piecewise-smooth path $\gamma: [a, b] \rightarrow \mathbb{R}^3$ that is bijective on $[a, b]$. The path γ is called a parametrization of C . A **simple closed curve** is the image of a piecewise-smooth path $\gamma: [a, b] \rightarrow \mathbb{R}^3$ that is bijective on $[a, b)$ and $\gamma(a) = \gamma(b)$.

Given a simple curve $C = \gamma([a, b])$, we can always orient it in one of two ways. If $\gamma'(t) \neq 0$ on $[a, b]$, i.e., γ is regular, then orientation is a continuous choice of the direction of the tangent vector. Going forward, all simple curves are assumed to

be oriented. We can now define path integrals along oriented curves by defining the oriented curve integral in terms of the associated path:

$$\int_C F \cdot ds = \int_\gamma F \cdot ds$$

If we denote by C^- the curve C with opposite orientation, then our prior work shows that

$$\int_{C^-} F \cdot ds = - \int_C F \cdot ds$$

We can talk about the algebraic sum of a finite number of oriented curves $C_1, \dots, C_k$.

$$C = C_1 + \dots + C_k$$

There is no restriction that the endpoints of the C_i agree. Then we can show that

$$\int_C F \cdot ds = \sum_{i=1}^k \int_{C_i} F \cdot ds$$

Exercise 4. Let C be the boundary of the square whose vertices are $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$, oriented counterclockwise. Evaluate each of the following line integrals around C :

- a) $\int_C (x + 2y)dx + (x - 3y^2)dy$
- b) $\int_C (x + y)dx + (x - 3y^2)dy$

2 Surface Integrals for Vector Fields

Just as we generalized path integrals for scalar functions to line integrals for vector fields, we can do a similar generalization of surface integrals for scalar functions to surface integrals for vector fields. As in the curve case, we need a notion of orientation.

Definition 5: Oriented Surface

A parametrized C^1 regular surface S is called **orientable** if a unit normal vector $\vec{n}$ is assigned at each point on S and $\vec{n}$ is continuous on S .

We remark that if a surface is orientable then it has two orientations: take the unit normal $-\vec{n}$ to obtain the surface S^- with opposite orientation. Some common examples of orientable surfaces and their normal vectors are discussed below.

If $\Phi: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a parametrization of an oriented regular surface S then the convention is to choose

$$\vec{n}(\Phi(u, v)) = \frac{\Phi_u \times \Phi_v}{\|\Phi_u \times \Phi_v\|}$$

If $S = \{(x, y, z) \mid G(x, y, z) = 0\}$ is a level surface for $G \in C^1, \nabla G \neq 0$ then S is orientable by picking

$$\vec{n}(x, y, z) = \frac{\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|}$$

If $S = \{(x, y, g(x, y)) \mid (x, y) \in D \subseteq \mathbb{R}^2\}$ is a graph then S is orientable with normal vector

$$\vec{n} = \frac{(-g_x, -g_y, 1)}{\sqrt{1 + g_x^2 + g_y^2}}$$

Definition 6: Surface Integral

Let S be an oriented surface in $\mathbb{R}^3$, and F a vector field defined on S . The **surface integral** of F over S is

$$\int_S F \cdot dS := \iint_S F \cdot \vec{n} dS$$

We have analogous reparametrization definitions and results.

Definition 7: Reparametrization

Suppose $\Phi: D \subseteq \mathbb{R}^2 \rightarrow S$ is a parametrization of S . A map $\tilde{\Phi}: \tilde{D} \subseteq \mathbb{R}^2 \rightarrow S$ is called a **reparametrization** of S if there is a bijective C^1 map $h: \tilde{D} \rightarrow D$ such that $h^{-1} \in C^1$ and

$$\tilde{\Phi}(s, t) = \Phi(h(s, t)) \quad \forall (s, t) \in \tilde{D}$$

A reparametrization is called **orientation-preserving** if

$$\frac{\tilde{\Phi}_s \times \tilde{\Phi}_t}{\|\tilde{\Phi}_s \times \tilde{\Phi}_t\|}(s, t) = \frac{\Phi_u \times \Phi_v}{\|\Phi_u \times \Phi_v\|}(h(s, t))$$

A reparametrization is called **orientation-reversing** if

$$\frac{\tilde{\Phi}_s \times \tilde{\Phi}_t}{\|\tilde{\Phi}_s \times \tilde{\Phi}_t\|}(s, t) = -\frac{\Phi_u \times \Phi_v}{\|\Phi_u \times \Phi_v\|}(h(s, t))$$

If the reparametrization is orientation-preserving we have

$$\iint_{\Phi(D)} F \cdot dS = \iint_{\tilde{\Phi}(\tilde{D})} F \cdot dS$$

If the reparametrization is orientation-reversing we have

$$\iint_{\Phi(D)} F \cdot dS = -\iint_{\tilde{\Phi}(\tilde{D})} F \cdot dS$$

In particular, if S is an oriented surface we have

$$\int_{S^-} F \cdot dS = -\int_S F \cdot dS$$

We conclude with a discussion about orientation of boundary surfaces. If piecewise-smooth $S \subseteq \mathbb{R}^3$ is contained in $\mathbb{R}^2$ (i.e., z -coordinate is zero on S), then we talk about its boundary ∂S having positive orientation such that a person walking along ∂S according to the positive direction will always have the inside of S to their left. Here the normal vector is $\hat{k} = (0, 0, 1)$. Unless stated otherwise the orientation of such boundaries is assumed to be positive.

In general, let $S \subseteq \mathbb{R}^3$ be a regular surface. We say $p \in S$ is an interior point of S if there is a local C^1 parametrization $\Phi: B_1(0) \subseteq \mathbb{R}^2 \rightarrow S$ with $\Phi(0) = p, \Phi_u(0) \times \Phi_v(0) \neq 0$. Any point where this condition fails is called a boundary point, and the collection of such points is denoted ∂S . For our purposes, ∂S will be a piecewise-smooth-regular curve. If S is oriented with normal $\vec{n}$, the positive orientation of ∂S is such that a person walking along ∂S with $\vec{n}$ up, the surface will always be on their left.

Exercise 5. Calculate $\int_S F \cdot dS$ where $F(x, y, z) = (x, y, z^4)$ where S is the surface of the hemisphere $x^2 + y^2 + z^2 = 9, z \geq 0$ including the disk $x^2 + y^2 \leq 9$ with $z = 0$. Assume S to be positively oriented.

These notes were prepared by Ariel Goodwin for MATH 248 at McGill University as taught by Pengfei Guan.

1 Green's Theorem and Stokes' Theorem

Before proceeding with Green's theorem, we recall the notion of induced positive orientation of a surface. This was discussed at the end of last week's tutorial, and the same discussion is repeated here.

In general, let $S \subseteq \mathbb{R}^3$ be a regular surface. We say $p \in S$ is an interior point of S if there is a local C^1 parametrization $\Phi: B_1(0) \subseteq \mathbb{R}^2 \rightarrow S$ with $\Phi(0) = p, \Phi_u(0) \times \Phi_v(0) \neq 0$. Any point where this condition fails is called a boundary point, and the collection of such points is denoted ∂S . For our purposes, ∂S will be a piecewise-smooth-regular curve. If S is oriented with normal $\vec{n}$, the positive orientation of ∂S is such that a person walking along ∂S with $\vec{n}$ up, the surface will always be on their left.

There is a special case that is used in the statement of Green's theorem. If a piecewise-smooth surface $S \subseteq \mathbb{R}^3$ is contained in $\mathbb{R}^2$ (i.e., z -coordinate is zero on S), then we talk about its boundary ∂S having positive orientation such that a person walking along ∂S according to the positive direction will always have the inside of S to their left. Here the normal vector is $\hat{k} = (0, 0, 1)$. Unless stated otherwise the orientation of such boundaries is assumed to be positive.

With that out of the way, we can present the elegant Green's theorem which is in some sense a version of the Fundamental Theorem of Calculus in two dimensions.

Theorem 1: Green's Theorem

Let $D \subseteq \mathbb{R}^2$ be a closed region such that ∂D is piecewise C^1 . If $\mathbf{F} = (P, Q)$ is a C^1 vector field on D then

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

You will frequently encounter this theorem in the following form:

$$\int_{\partial D} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

This is a powerful result that allows us to transform some tricky path integrals into double integrals, or vice versa. It is remarkable that in some sense all of the information conveyed in doing a double integral over the entire region D is equivalent to the information carried by a path integral along the boundary.

Exercise 1. Let C be the positively oriented path along the triangle with vertices $(0, 0), (2, 0), (2, 2)$. Compute

$$\int_C x \sin(x^2) dx + (3e^{y^2} - 2x) dy$$

(Answer: -4 , apply Green's theorem.)

Exercise 2. Let C be a positively oriented circle of radius 1 around the origin, and suppose $\phi(x), \psi(y)$ are two smooth functions on $\mathbb{R}$. Which of the following integrals are necessarily zero?

- a) $\int_C \phi(y)dx + \psi(x)dy$
- b) $\int_C \phi(xy)ydx + \phi(xy)xdy$
- c) $\int_C \phi(x)\psi(y)dy$

(Answer: Only (b) is necessarily zero. Apply Green's theorem and come up with simple counterexamples for (a) and (c).)

Let's examine some of the nice corollaries.

Theorem 2: Area Formula

Let $D \subseteq \mathbb{R}^2$ be a closed region such that ∂D is piecewise C^1 . Then

$$A(D) = \int_{\partial D} xdy = - \int_{\partial D} ydx = \frac{1}{2} \int_{\partial D} xdy - ydx$$

Theorem 3: Difference of Regions

Let $D_1, D_2 \subseteq \mathbb{R}^2$ be closed regions such that $\partial D_1, \partial D_2$ are piecewise C^1 , with orientations induced by D_1, D_2 respectively. Set $D = D_1 \setminus D_2$. If $\mathbf{F} = (P, Q)$ is a C^1 vector field on D such that $Q_x = P_y$ on D then

$$\int_{\partial D_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial D_2} \mathbf{F} \cdot d\mathbf{s}$$

Theorem 4: Integration by Parts

Let $D \subseteq \mathbb{R}^2$ be a closed region such that ∂D is piecewise C^1 . Let $\phi, \psi \in C^2(D)$. Then

$$\begin{aligned} \iint_D \phi \Delta \psi dA &= - \iint_D \nabla \phi \cdot \nabla \psi dA + \int_{\partial D} \phi \nabla \psi \cdot \mathbf{n} ds \\ \iint_D \phi \Delta \psi - \psi \Delta \phi dA &= \int_{\partial D} (\phi \nabla \psi - \psi \nabla \phi) \cdot \mathbf{n} ds \end{aligned}$$

You may also encounter Green's theorem in a so-called vector form, which can be obtained by identifying $\mathbb{R}^2$ within $\mathbb{R}^3$ as a space with z -coordinate zero. More precisely, assuming all the hypotheses of Green's theorem, treat $\mathbf{F} = (P, Q)$ as $\mathbf{F} = (P, Q, 0)$ and we find

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D (\text{curl} \mathbf{F}) \cdot \mathbf{k} dA$$

In the same vein, we can let $\mathbf{n}$ be the outer normal of ∂D and obtain a divergence form of Green's theorem:

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{n} ds = \iint_D \operatorname{div} \mathbf{F} dA$$

Exercise 3. Verify the divergence theorem for $\mathbf{F}(x, y) = (x, y)$ on the closed unit disk D . Then evaluate the integral of the normal component of $\mathbf{G}(x, y) = (2xy, -y^2)$ around the ellipse $x^2/a^2 + y^2/b^2 = 1$. (Answer: Verify that both sides of the divergence form of Green's theorem are equal in this case, noticing that the unit normal vector $\mathbf{n}$ to ∂D is given by $\mathbf{n}(x, y) = (x, y)$. For the second part, use the divergence form of Green's theorem. Answer is zero.)

Our next spectacular result is Stokes' theorem. It generalizes Green's theorem to three dimensions, and we can even derive Green's theorem from it (although the proof you might see uses Green's theorem).

Theorem 5: Stokes' Theorem

Let $S \subseteq \mathbb{R}^3$ be a compact regular oriented surface with positively oriented piecewise C^2 boundary ∂S . Let $\mathbf{F}$ be a C^1 vector field on S . Then

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{dS} = \int_{\partial S} \mathbf{F} \cdot \mathbf{ds}$$

Exercise 4. Evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{dS}$ where $\mathbf{F} = (z^2, -3xy, x^3y^3)$ and S is the surface defined by the portion of the graph $z = 5 - x^2 - y^2, z \geq 1$, with positive orientation. (Answer: 0, draw a picture of the surface and notice that the boundary is $\{(x, y, 1) \mid x^2 + y^2 = 2\}$, then apply Stokes' theorem.)

Exercise 5. Evaluate the surface integral $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{dS}$ where S is the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$ and $\mathbf{F}(x, y, z) = (x^3, -y^3, 0)$. (Answer: 0, draw a picture of the surface and notice that the boundary is $\{(x, y, 0) \mid x^2 + y^2 = 1\}$, then apply Stokes' theorem.)

For many analogous reasons to Green's theorem, this result is remarkable and has a number of interesting corollaries.

Theorem 6: Common Boundary

Let $S_1, S_2 \subseteq \mathbb{R}^3$ be two oriented surfaces satisfying the conditions of Stokes' theorem such that $\partial S_1 = \partial S_2 = \gamma$. Suppose also that S_1, S_2 induce opposite orientations on γ . Then for any C^1 $\mathbf{F}$:

$$\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot \mathbf{dS} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot \mathbf{dS}$$

Theorem 7: No Boundary

If S is a compact oriented surface satisfying the conditions of Stokes' theorem, and S is without boundary, then

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$$

2 Gauss's Divergence Theorem

The last big theorem of vector calculus is the divergence theorem. It gives us a relationship between triple integrals and surface integrals. Suppose $W \subseteq \mathbb{R}^3$ is a bounded solid region with piecewise-smooth boundary $S = \partial W$. We assign an orientation on S so that its normal vector points outward.

Theorem 8: Gauss' Divergence Theorem

Suppose $W \subseteq \mathbb{R}^3$ is a bounded solid region with piecewise-smooth boundary $S = \partial W$. Let $\mathbf{F}$ be a C^1 vector field on W . Then

$$\iiint_W \operatorname{div} \mathbf{F} dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

Exercise 6. Let S be the surface of the cube whose main diagonal has endpoints $(0, 0, 0)$ and $(1, 1, 1)$. Let $\mathbf{F}(x, y, z) = (z, y, x)$. What is the value of the flux of $\mathbf{F}$ through S , given that S has outward orientation? (Recall that flux is another term for surface integral over closed surface.) (Answer: 1, draw a picture and apply the divergence theorem, making use of fact that $\iiint_W dV = \operatorname{Vol}(W)$.)

We study some of the corollaries.

Theorem 9: Difference of Surfaces

Suppose $W \subseteq \mathbb{R}^3$ is a bounded solid region satisfying the assumptions of Gauss's theorem, and that $\partial W = S_1 - S_2$. If $\operatorname{div} \mathbf{F} = 0$ on W then

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$$

Oftentimes our field $\mathbf{F}$ will have undesirable behaviour at certain isolated points, for example division by zero causing blowup. Such points are called singularities, and even if we cannot apply Gauss's theorem directly we can often make use of some results on singularities to manage the problem.

Theorem 10: Singularities

Suppose $W \subseteq \mathbb{R}^3$ is a bounded solid region satisfying the assumptions of Gauss's theorem. Let $p_1, \dots, p_N \subseteq \text{int } W$. Suppose $\mathbf{F}$ is a C^1 vector field on $W \setminus \{p_1, \dots, p_N\}$. Let $\delta > 0$ sufficiently small so that $B_\delta(p_i) \subseteq \text{int } W$ for all $i = 1, \dots, N$. Then

$$\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \sum_{i=1}^N \iint_{\partial B_\delta(p_i)} \mathbf{F} \cdot d\mathbf{S}$$

The usual idea is to determine your singularities, isolate them in a number of small balls of radius $\delta > 0$ for some arbitrary δ sufficiently small, and apply Gauss's theorem away from the singularity. Then try to control the behaviour of the leftover integrals as $\delta \rightarrow 0$ (remember we chose δ arbitrarily so we are free to vary it!).

Exercise 7. Let $\mathbf{F}$ be a C^1 vector field on $\mathbb{R}^3$. Prove that

$$\text{div} \mathbf{F}(0) = \lim_{r \rightarrow 0} \frac{3 \int_{\partial B_r(0)} \mathbf{F} \cdot d\mathbf{S}}{4\pi r^3}$$

(Solution: Notice that $3/(4\pi r^3) = 1/\text{Vol}(B_r(0))$. Then consider the difference

$$\left| \frac{1}{\text{Vol}(B_r(0))} \int_{\partial B_r(0)} \mathbf{F} \cdot d\mathbf{S} - \text{div} \mathbf{F}(0) \right|$$

Apply the divergence theorem and use the continuity of $\text{div} \mathbf{F}$ to show that the difference goes to zero as $r \rightarrow 0$.)

These notes were prepared by Ariel Goodwin for MATH 248 at McGill University as taught by Pengfei Guan.

1 Conservative Vector Fields

Recall that a C^1 vector field $\mathbf{F}: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ is conservative if there exists a C^1 function $f: \Omega \rightarrow \mathbb{R}$ such that $\mathbf{F}(x) = \nabla f(x)$ for all $x \in \Omega$. Conservative vector fields have many nice properties that we already know: line integrals of conservative vector fields obey a fundamental theorem of calculus (in particular, they depend only on the endpoints of the curve), and their curl is zero (whenever curl is defined, e.g. $n = 2, 3$). Note that the curl of a vector field $\mathbf{F} = (P, Q)$ on $\mathbb{R}^2$ is taken to be $Q_x - P_y$ (reminds us of Green's theorem).

This gives us some methods for checking if a vector field is not conservative (non-zero curl or two distinct values for line integrals with different curves but the same endpoints). We have an example showing that being conservative is not equivalent to zero curl:

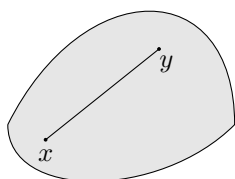
$$\mathbf{F}(x, y, z) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right), \quad \forall (x, y, z) \in \mathbb{R}^3 \setminus \mathbb{R}_z$$

where $\mathbb{R}_z$ is the z -axis. It would be nice if zero curl implied conservative. It turns out that for a large class of vector fields defined on certain regions that this is the case.

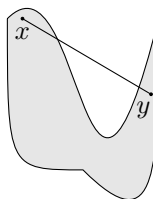
Definition 1: Convexity

A set $C \subseteq \mathbb{R}^n$ is **convex** if $\lambda x + (1 - \lambda)y \in C$ for all $x, y \in C$, $\lambda \in [0, 1]$.

This definition says that for any two points x, y in a convex set C , the line connecting x to y is contained in C . Convex sets are very natural, and drawing some pictures allows us to get a feel for them.



Convex set



Non-convex set

As an aside, convexity is a simple notion with many deep consequences throughout mathematics. The study of convex sets and functions forms the basis of convex analysis, a rich geometric tool that finds much use in optimization, for example.

Theorem 1: Conditions for Conservative Fields

Let $\Omega \subseteq \mathbb{R}^3$ be convex, let $W = \Omega \setminus \{p_1, \dots, p_k\}$ where $p_1, \dots, p_k$ is a finite collection of points in Ω and suppose $\mathbf{F}: W \rightarrow \mathbb{R}^3$ is C^1 . Then the following are equivalent:

- a) For any oriented simple closed curve $\gamma \subseteq W$, $\int_{\gamma} \mathbf{F} \cdot d\mathbf{s} = 0$.
- b) $\int_{\gamma_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\gamma_2} \mathbf{F} \cdot d\mathbf{s}$ for any oriented simple curves $\gamma_1, \gamma_2 \subseteq W$ that share the same endpoints.
- c) $\mathbf{F}$ is conservative, i.e., $\mathbf{F} = \nabla f$ for some C^1 function f on W .
- d) $\text{curl } \mathbf{F} = 0$

We remark that this theorem holds more generally if Ω is assumed only to be *simply connected* rather than convex, which is significantly weaker. The topological notion of simple connectivity is beyond the scope of this course, however.

Exercise 1. Determine if the following vector fields are conservative, and if so find a corresponding f .

- a) $\mathbf{F}(x, y) = (\cos xy - xy \sin xy, -x^2 \sin xy)$
- b) $\mathbf{F}(x, y) = (x\sqrt{x^2y^2 + 1}, y\sqrt{x^2y^2 + 1})$
- c) $\mathbf{F}(x, y) = (2x \cos y + \cos y, -x^2 \sin y - x \sin y)$

Exercise 2. Let $\mathbf{F}(x, y, z) = (2xyz + \sin x, x^2z, x^2y)$. Find a function f such that $\mathbf{F} = \nabla f$.

Exercise 3. Suppose $\mathbf{F} = (F_1, F_2, F_3)$ where $F_i(tx, ty, tz) = tF_i(x, y, z)$ for all $t \in \mathbb{R}$, $(x, y, z) \in \mathbb{R}^3$. Suppose also that $\text{curl } \mathbf{F} = 0$. Prove that $\mathbf{F} = \nabla f$, where

$$f(x, y, z) = \frac{1}{2} (xF_1(x, y, z) + yF_2(x, y, z) + zF_3(x, y, z))$$

Hint: Use Assignment 2, Q4.

Exercise 4. Let $\mathbf{F}(x, y, z) = (e^x \sin y, e^x \cos y, z^2)$. Compute $\int_{\gamma} \mathbf{F} \cdot d\mathbf{s}$ where $\gamma(t) = (\sqrt{t}, t^3, e^{\sqrt{t}})$ for $0 \leq t \leq 1$.

Exercise 5. Prove that if $\mathbf{F} = (F_1, F_2, F_3)$ is a C^1 vector field on $\mathbb{R}^3$ with $\text{div } \mathbf{F} = 0$ then there exists a C^1 vector field $\mathbf{G}$ with $\mathbf{F} = \text{curl } \mathbf{G}$. Hint: Define $\mathbf{G} = (G_1, G_2, G_3)$ by

$$\begin{aligned} G_1(x, y, z) &= \int_0^z F_2(x, y, t) dt - \int_0^y F_3(x, t, 0) dt \\ G_2(x, y, z) &= - \int_0^z F_1(x, y, t) dt \\ G_3(x, y, z) &= 0 \end{aligned}$$